

SUGGESTED EXPERIMENTS IN SCHOOL MATHEMATICS

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1

Patterns in Numbers

Experiments : 101 to 110

Experiment 101. Patterns in Rows and Columns.

Let the teacher write all the numbers from 1 to 100 on the blackboard and ask the children to find all the patterns they find there :

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Some patterns which they may discover are :

- (i) All numbers in the first column end with 1, all numbers in the second column end with 2 and so on till all numbers in the tenth column end with 0.

- (ii) If we leave out the last column, all numbers in the second row begin with 1, all numbers in the third row begin with 2 and so on till all numbers in the tenth row begin with 9.
- (iii) Numbers in each row increase by unity.
- (iv) Numbers in each column increase by ten.

After this, children can be asked questions like the following :

- (a) Suppose the table of numbers is continued beyond 100, in which columns and rows will the following numbers occur : 3456, 2347, 4589, 3250 ?
- (b) What are the sums of numbers in various rows and columns ? Do you see any patterns there ?

Experiment 102. Patterns in Diagonals.

- (i) Successive numbers in diagonals shown by dotted lines differ by 9. The numbers increase successively by 9 if we go down a diagonal and they decrease successively by 9 if we go up a diagonal. A sequence of numbers in which numbers increase or decrease by a constant number is called an *arithmetic progression*. Children can give other examples e.g., numbers in each row form an arithmetic progression with common difference 1 and numbers in each column form an A.P. with common difference 10. If we consider numbers in first two columns and add the numbers along the dotted lines, they form an A.P. with common difference 20. Children can also find A.P.'s here with common difference 30 or 40 or 50 etc. The sequence of sums of numbers in rows (or columns) also form an A.P.
- (ii) Successive numbers in diagonals shown by full lines differ by 11. The numbers increase successively by 11 as we go down a diagonal and they decrease successively by 11 as we go up a diagonal.
- (iii) If we leave out the last column, the sum of the digits of numbers along a dotted diagonal is the same. However if when the sum of digits exceeds 9, we again take the sum of digits, we have not even to leave the last column.
- (iv) If we leave out the last column, the sums of digits of numbers along a full-line diagonal form an A.P. with common difference 2.

- (v) Children can form other diagonals like those shown by dotted lines and study their patterns in the same way as above :

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

- (vi) Children can be asked questions like the following :

- (a) Form diagonals in which the difference is 29, 31, 41 etc.
- (b) Find the sum of all numbers in each diagonal.

Experiment 103. Other Similar Patterns.

- (i) Let the children find similar patterns in rows, columns and diagonals of the following two tables.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49
50	51	52	53	54	55	56

- (ii) Let the children compare the patterns in diagonals of all the three tables given so far. The dotted diagonals form A.P.'s with common difference 9, 5 and 6 respectively. The full-line diagonals form A.P.'s with common difference 11, 7 and 8 respectively. If they form similar dotted and full-line diagonals in similar table with 4, 5, 8 or 9 columns, what common differences do they expect?

It is obvious that the above tables can give a really rich variety of patterns and the patterns we have given above form just a sample. The children may be encouraged to find more patterns and to try to understand the reasons for the same.

Experiment 104. Patterns in Multiples of Two, Five and Ten.

- (i) Let the children write all numbers 1 to 100 as in the first table and enclose multiples of 2 in full circles¹, multiples of 5 in dotted circles and multiples of 10 in full squares. Different coloured circles may also be used for this purpose.

1. For convenience, in the following figures, we have drawn closed oval curves (ellipses) instead of circles and rectangles instead of squares.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Let the children note that :—

- all multiples of 2 [denoted by $M(2)$] occur in even-numbered columns.
- all multiples of 5 [denoted by $M(5)$] occur in columns whose numbers are multiples of 5.
- all multiples of 10 [denoted by $M(10)$] occur in columns whose number is 10.
- if one number in a column is included, each number in that column is included.

Experiment 105. Patterns in Multiples of Three and Nine.

In the following figure, multiples of 3 are enclosed in full circles and multiples of 9 are enclosed in dotted circles.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Let the children note that :

- (i) Multiples of 3 [denoted by $M(3)$] show a diagonal patterns and multiples occur in every column.
- (ii) Multiples of 9 [denoted by $M(9)$] also show a diagonal pattern and multiples occur in every column.
- (iii) The number of multiples of 3 in successive rows are 3, 3, 4, 3, 3, 4, 3, 3, 4, The same pattern holds for multiples of 3 in successive columns. How many multiples of 3 will be in the 20th row, 21st row, 22nd row ?
- (iv) The number of multiples of 9 in successive rows are 1, 1, 1, 1, 1, 1, 2, 1, . . . What will be the pattern if we continue the numbers beyond 100 ?
- (v) In every row (or column) there are two non-multiples of 3 between every two multiples of 3.

Experiment 106. Patterns in Multiples of Four and Eight.

In the following figure, multiples of 4 are enclosed in full circles and multiples of 8 are enclosed in dotted circles.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Let the children note that :

- (i) Multiples of 4 and 8 show diagonal pattern, but all columns are not represented.
- (ii) The numbers of multiples of 4 in successive rows are 2, 3, 2, 3, 2, 3, 2, 3, and the numbers of multiples of 4 in successive columns are 0, 5, 0, 5, 0, 5, 0, 5, 0, 5,

- (iii) The numbers of multiples of 8 in successive rows are 1, 1, 1, 2, 1, 1, 1, 2, 1, 1,

The numbers of multiples of 8 in successive rows are 0, 2, 0, 2, 0, 3, 0, 3, 0, 2,

Experiment 107. Patterns in Multiples of Six and Seven.

In the following figure, multiples of 6 are enclosed in full circles and multiples of 7 are enclosed in dotted circles.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Let the children note that :

- (i) Multiples of 6 show a diagonal pattern, but all columns are not represented.
- (ii) Multiples of 7 show a diagonal pattern, but all columns are represented.

Experiment 108. Summary of Patterns in Multiples.

Let the children summarise the results of the last four experiments as follows :

- (a) $M(2)$, $M(5)$, $M(10)$ show column patterns. Some columns are represented completely, others are not represented at all.
- (b) $M(3)$, $M(7)$, $M(9)$ show diagonal pattern and all columns are represented.

- (c) $M(4)$, $M(6)$, $M(8)$ show diagonal patterns and all columns are not represented.

What is special about these three different cases? We note that:

- (a) 2, 5, 10 are factors of 10
- (b) 3, 7, 9 are not factors of 10, nor they have a factor common with 10. These numbers are said to be prime to 10.
- (c) 4, 6, 8 are not factors of 10, but they have a factor common with 10

What patterns do the children expect when they take multiples of 11 or 12 or 13 etc.?

- (i) Let the children circle $M(2)$ and $M(3)$ in different colours and note that $M(6)$ are circled in both colours. Similarly let them circle $M(2)$, $M(3)$ and $M(4)$ with different colours and note that $M(12)$ are circled in all three colours. What conclusion do they draw?
- (ii) Instead of drawing circles, the children may write the numbers in squares and shade them as in the following figure which shows multiples of 11 and 12 and shows the diagonal patterns.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

- (iii) An alternative way is to make 'window-readers' for $M(2)$, $M(3)$, $M(4)$, ..., $M(10)$, $M(11)$, $M(12)$ etc. These are card board pieces of the same size as the above big

square from which squares representing $M(2)$, $M(3)$, etc. have been cut out. When a reader is placed above the complete 10×10 square, it shows the corresponding multiples. When readers of $M(2)$, $M(3)$ are placed simultaneously, $M(6)$ are shown. The original 10×10 square with numbers 1 to 100 may be made on a flannel graph [see Chapter 14].

Experiment 109. Patterns in Multiples with Nine or Less Columns.

We now see whether a similar pattern holds when we arrange numbers in 9 or less columns.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42

It can be verified that

- (i) multiples of 2, 3, 6 form column patterns.
- (ii) multiples of 5 show a diagonal pattern in which all columns are represented.
- (iii) multiples of 4 show a diagonal pattern in which all columns are not represented.

In the same way let the children arrange numbers in 7, 8, 9, 11, 12,..... columns and note that the above results continue to hold viz.

- (i) multiple of factors of the number of columns show column pattern.
- (ii) multiple of those numbers which are prime to the number of columns show diagonal patterns in which all columns are represented.
- (iii) multiple of other numbers show diagonal patterns in which all columns are not represented.

Experiment 110. Patterns in Primes

- (i) Let the children find all the prime numbers less than 500 (say). These are :

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53,
59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113,
127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181,
191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251,
257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317,
331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397,
401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463,
467, 479, 487, 491, 499.

- (ii) Let the children arrange these in 10×10 , 15×15 , 20×20 , 25×25 , 30×30 , squares and mark with dots the prime numbers therein and try to find if there are any patterns. They may note that some columns do not contain any prime numbers and some contain only one. What is the reason? Let them also count the number of primes in each row.

- (iii) Let the children form the following table :

N	Number of primes less than N
50	
100	
150	
200	
250	
300	
...	
...	
...	
...	
1000	

There is a pattern here, though the children may find it difficult to find it, since it involves logarithmic function.

- (iv) Let the children find the number of twin primes *i.e.*, prime pairs differing by 2 only *e.g.*, (3, 5), (5, 7), (11, 13), (17, 19), (29, 31) etc. below any given number.
- (v) Let the children guess how many prime numbers are there in all and how many twin primes there are in all. It has been proved that the number of prime numbers is infinite. It is also believed that the number of twin primes is also infinite, though the result has not been proved so far. The children may be interested to know that the largest *known* prime number is $2^{4423}-1$ which has 1332 digits and the largest *known* twin primes are :

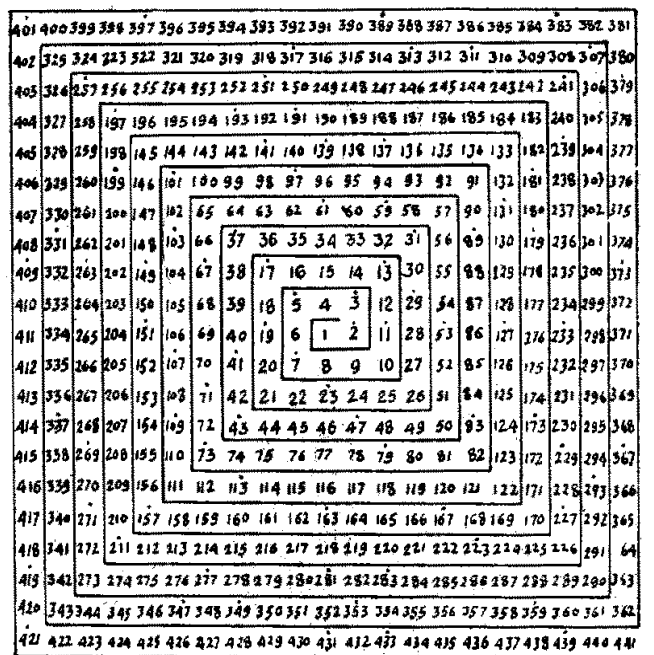
140,737,488,353,699 and 140,737,488,353,701.

The children may wonder how it has been possible to find these. The secret is in the electronic computer and the mind of man.

- (vi) Let the children continue the spiral pattern shown on the next page.

Do the children find any patterns here? Do they find that some diagonals contain even numbers only and some contain odd numbers only?

In similar 10×10 , 15×15 , 20×20 , 25×25 spiral patterns, let the children mark the primes with dots and see what patterns they get.



Patterns in Squares, Cubes and other Powers of Natural Numbers

Experiments : 111 to 120

Experiment 111. Preparation of Tables of Squares and Cubes

Let the children prepare the following table :

Number	Square	Cube	Number	Square	Cube	Number	Square	Cube
1	1	1	18	324	5832	35	1225	42875
2	4	8	19	361	6859	36	1296	46656
3	9	27	20	400	8000	37	1369	50653
4	16	64	21	441	9261	38	1444	54872
5	25	125	22	484	10648	39	1521	59319
6	36	216	23	529	12167	40	1600	64000
7	49	433	24	576	13824	41	1681	68921
8	64	512	25	625	15625	42	1784	74928
9	81	729	26	676	17576	43	1849	79507
10	100	1000	27	729	19683	44	1936	85184
11	121	1331	28	784	21952	45	2025	91125
12	144	1728	29	841	24389	46	2116	97336
13	169	2197	30	900	27000	47	2209	103823
14	196	2744	31	961	29171	48	2304	110592
15	225	3375	32	1024	32768	49	2401	117649
16	256	4096	33	1089	35937	50	2500	125000
17	289	4913	34	1156	39304			

The children can form teams for this purpose and the whole class can combine together to prepare a correct table of squares and cubes of all numbers upto 100. There can be a competition as to who makes the least number of mistakes.

Experiment 112. Patterns in Squares and Cubes of odd and even numbers

There are two types of natural numbers viz. even and odd. Even numbers are numbers like 2, 4, 6, 8,... which are divisible by 2. Odd numbers are those like 1, 3, 5, 7,... which are not divisible by 2.

Let the children note that :

- (i) Squares of even numbers are always even and further squares of even numbers are multiples of 4 (every even number has 2 as a factor, therefore its square has 4 as a factor)
- (ii) squares of odd numbers are always odd.
- (iii) cubes of even numbers are even and are multiples of eight.
- (iv) cubes of odd numbers are always odd.

Experiment 113. Patterns in Powers of Numbers of three Classes Obtained by Dividing Numbers by three.

On dividing a natural number by 3, we get three types of numbers.

- (a) Numbers of the first kind which give 0 as remainder e.g. 3, 6, 9, 12, 15,.....

These are of the form $3n$ where n is any natural number.

- (b) numbers of the second kind which give 1 as remainder e.g. 4, 7, 10, 13, 16,.....

These are of the form $3n+1$.

- (c) numbers of the third kind which give 2 as remainder e.g. 5, 8, 11, 14, 17,.....

These are of the form $3n+2$.

The children easily find that :

- (i) squares of numbers of the first kind are numbers of the first kind and are multiples of 9 [$(3n)^2 = 9n^2$]
- (ii) squares of numbers of the second kind are numbers of the second kind [$(3n+1)^2 = 3(3n^2+2n)+1$]
- (iii) squares of numbers of the third kind are numbers of the second kind [$(3n+2)^2 = 3(3n^2+4n+1)+1$]

- (iv) no number of the third kind can be a square number.
- (v) the cube of a number of the first kind is a number of the first kind and is a multiple of 27.
 $[(3n)^3 = 3 \times 9n^3 = 27n^3]$
- (vi) the cube of a number of the second kind is a number of the second kind.
 $[(3n+1)^3 = 27n^3 + 27n^2 + 9n + 1 = 3(9n^3 + 9n^2 + 3n + 4) + 1]$
- (vii) the cube of a number of the third kind is a number of the third kind.
 $[(3n+2)^3 = 27n^3 + 54n^2 + 36n + 8 = 3(9n^3 + 18n^2 + 12n + 2) + 2]$
 The children can see these results more easily by arranging numbers as below :

First Kind : A	Second Kind : B	Third Kind : C
3	1	2
6	4	5
9	7	8
12	10	11
15	13	14
18	16	17
21	19	20
24	22	23
27	25	26
30	28	29
33	31	32

and then seeing that the square of any number in column A is in A, the square of any number in column B is in B, the square of any number in column C is in B, the cube of any number in column A is in A, the cube of any number of column B is in B and the cube of any number in column C is in C. Symbolically we can write

$$A^2=A, B^2=B, C^2=B, A^3=A, B^3=B, C^3=C$$

Similarly, the children can show that :

$$A^4=A, B^4=B, C^4=B, A^5=A, B^5=B, C^5=C$$

They can now deduce the general results :

- (i) Every power of a number in A is in A
- (ii) Every power of a number in B is in B
- (iii) Odd powers of a number in C are in C and even powers of a number in C are in B.

Experiment 114. Patterns in Powers of Numbers of Four Classes Obtained by Dividing Numbers by Four.

In the same way children can divide numbers into four classes viz. these which leave remainders 0, 1, 2, 3 respectively on being divided by 4, so that we get the following classes :

D	E	F	G
4	1	2	3
8	5	6	7
12	9	10	11
16	13	14	15
20	17	18	19
24	21	22	23
28	25	26	27
...	...	...	...
...	...	...	...

The children can now prove the results

$$D^2=D, E^2=E, F^2=D, G^2=E$$

$$D^3=D, E^3=E, F^3=D, G^3=G$$

$$D^4=D, E^4=E, F^4=D, G^4=E$$

$$\dots\dots\dots$$

and then obtain the general results :

- (i) all powers of numbers in class D are in D
- (ii) all powers of numbers in class E are in E
- (iii) all powers of numbers in class F are in D
- (iv) even powers of number in G are in E and odd powers of numbers in G are in G.

Experiment 115. Patterns in Powers of Numbers of Five Classes Obtained by Dividing Numbers by Five.

By dividing numbers by 5, we get the following classes :

P	Q	R	S	T
5	1	2	3	4
10	6	7	8	9
15	11	12	13	14
20	16	17	18	19
25	21	22	23	24
30	26	27	28	29
35	31	32	33	34
...	...	...	...	...
...	...	...	...	...

The children can show that

$$\begin{aligned} P^2 &= P, & Q^2 &= Q, & R^2 &= T, & S^2 &= T, & T^2 &= Q \\ P^3 &= P, & Q^3 &= Q, & R^3 &= S, & S^3 &= R, & T^3 &= T \\ P^4 &= P, & Q^4 &= Q, & R^4 &= Q, & S^4 &= Q, & T^4 &= Q \\ P^5 &= P, & Q^5 &= Q, & R^5 &= R, & S^5 &= S, & T^5 &= T \\ &\dots\dots\dots, & \dots\dots\dots, & \dots\dots\dots, & \dots\dots\dots, & \dots\dots\dots \end{aligned}$$

Thus

- (i) all powers of numbers in P belong to P
- (ii) all powers of numbers in Q belong to Q
- (iii) successive powers of numbers in R belong to R, T, S, Q, R,
- (iv) successive powers of numbers in T belong to T, Q, T, Q,
- (v) First, fifth, ninth, thirteenth, powers of numbers in any column belong to that column.

Experiment 116. Patterns in Unit Digits of Powers of Numbers

Children can easily prepare the following table :

Unit digit in number	0	1	2	3	4	5	6	7	8	9
Unit digit in square	0	1	4	9	6	5	6	9	4	1
Unit digit in cube	0	1	8	7	4	5	6	3	2	9
Unit digit in fourth power	0	1	6	1	6	5	6	1	6	1
Unit digit in fifth power	0	1	2	3	4	5	6	7	8	9

The children can easily deduce the following results :

- (i) square or sixth power or tenth power of no number etc. can end with 2, 3, 7, 8
- (ii) fourth power or eighth power or twelfth power etc. of no number can end with 2, 3, 4, 7, 8, 9
- (iii) unit digit of 5th power, 9th power and 13th power of a number is the same as the unit digit of the original number.
- (iv) If we know the unit digit of the third power or of 5th power or of 7th power, we can tell the unit digit of the original number uniquely.
- (v) If unit digit of a perfect square is 1, the unit digit of the original number may be 1 or 9, if the unit digit of the square is 4, the number's unit digit may be 2 or 8 and so on.

- (vi) If the unit digit of a perfect fourth power is 1, the unit digit in the original number may be 1 or 3 or 7 or 9 and if the unit digit in the perfect fourth power is 6, the unit digit in the original number is 2, 4, 6 or 8.

Experiment 117. Patterns in Number of Digits in Powers of Numbers

Children can prepare the following table :

Number of digits in the number :	1	2	3	4	5
Number of digits in its square :	1,2	3,4	5,6	7,8	9,10
Number of digits in its cube :	1,2,3	4,5,6	7,8,9	10,11,12	13,14,15
Number of digits in its fourth power	1,2,3,4	5,6,7,8	10,11,12	13,14,15,16	17,18,19,20

The children may be able to see the pattern that if an m digit number is raised to power n , the largest number of digits, in the n th power of an m digit number is $m \times n$ and the least number is $(m-1)n + 1$.

The children can now answer questions like the following :

If the square of a number consists of 17 (or 18 or 19 or 30) digits, how many digits does the number have ?

Thus if we know the number of digits in a given power of a number, can we find the number of digits in the number uniquely ?

Experiment 118. Expressing Numbers as Sums of Four or Less Square Numbers

Ask the children to express each number as the sum of four or less square numbers in as many ways as they can. Let them collectively prepare the following table :

$1 = 1^2$	$2 = 1^2 + 1^2$
$3 = 1^2 + 1^2 + 1^2$	$4 = 1^2 + 1^2 + 1^2 + 1^2 = 2^2$
$5 = 2^2 + 1^2$	$6 = 2^2 + 1^2 + 1^2$
$6 = 2^2 + 1^2 + 1^2$	$7 = 2^2 + 1^2 + 1^2 + 1^2$
$8 = 2^2 + 2^2$	$9 = 2^2 + 2^2 + 1^2 = 3^2$
$10 = 2^2 + 2^2 + 1^2 + 1^2 = 3^2 + 1^2$	$11 = 3^2 + 1^2 + 1^2$
$12 = 3^2 + 1^2 + 1^2 + 1^2 = 2^2 + 2^2 + 2^2$	$13 = 3^2 + 2^2 = 2^2 + 2^2 + 2^2 + 1^2$
$14 = 3^2 + 2^2 + 1^2$	$15 = 3^2 + 2^2 + 1^2 + 1^2$
$16 = 4^2 = 2^2 + 2^2 + 2^2 + 2^2$	$17 = 4^2 + 1^2 = 3^2 + 2^2 + 2^2$
$18 = 3^2 + 3^2 = 4^2 + 1^2 + 1^2 = 3^2 + 2^2 + 2^2 + 1^2$	

$$\begin{aligned}
19 &= 3^2 + 3^2 + 1^2 = 4^2 + 1^2 + 1^2 + 1^2 & 20 &= 4^2 + 2^2 = 3^2 + 3^2 + 1^2 + 1^2 \\
21 &= 4^2 + 2^2 + 1^2 = 3^2 + 2^2 + 2^2 + 2^2 & 22 &= 4^2 + 2^2 + 1^2 + 1^2 = 3^2 + 3^2 + 2^2 \\
23 &= 3^2 + 3^2 + 2^2 + 1^2 & 24 &= 4^2 + 2^2 + 2^2 \\
25 &= 5^2 = 4^2 + 2^2 + 2^2 + 1^2 = 4^2 + 3^2 \\
26 &= 5^2 + 1^2 = 3^2 + 3^2 + 2^2 + 2^2 = 4^2 + 3^2 + 1^2 \\
27 &= 5^2 + 1^2 + 1^2 = 3^2 + 3^2 + 3^2 = 4^2 + 3^2 + 1^2 + 1^2 \\
28 &= 5^2 + 1^2 + 1^2 + 1^2 = 3^2 + 3^2 + 3^2 + 1^2 = 4^2 + 2^2 + 2^2 + 2^2 \\
29 &= 5^2 + 2^2 = 4^2 + 3^2 + 2^2 & 30 &= 5^2 + 2^2 + 1^2 = 4^2 + 3^2 + 2^2 + 1^2 \\
31 &= 5^2 + 2^2 + 1^2 + 1^2 = 3^2 + 3^2 + 3^2 + 2^2 \\
32 &= 4^2 + 4^2 \\
33 &= 4^2 + 4^2 + 1^2 = 4^2 + 3^2 + 2^2 + 2^2 = 5^2 + 2^2 + 2^2 \\
34 &= 4^2 + 4^2 + 1^2 + 1^2 = 4^2 + 3^2 + 3^2 = 5^2 + 2^2 + 2^2 + 1^2 \\
35 &= 5^2 + 3^2 + 1^2 = 4^2 + 3^2 + 3^2 + 1^2 \\
36 &= 6^2 = 3^2 + 3^2 + 3^2 + 3^2 = 5^2 + 3^2 + 1^2 + 1^2 = 4^2 + 4^2 + 2^2 \\
37 &= 6^2 + 1^2 = 5^2 + 2^2 + 2^2 + 2^2 = 4^2 + 4^2 + 2^2 + 1^2 \\
38 &= 6^2 + 1^2 + 1^2 = 5^2 + 3^2 + 2^2 = 4^2 + 3^2 + 3^2 + 2^2 \\
39 &= 6^2 + 1^2 + 1^2 + 1^2 = 5^2 + 3^2 + 2^2 + 1^2 \\
40 &= 6^2 + 2^2 = 4^2 + 4^2 + 2^2 + 2^2 \\
41 &= 6^2 + 2^2 + 1^2 = 4^2 + 4^2 + 3^2 = 5^2 + 4^2 \\
42 &= 6^2 + 2^2 + 1^2 + 1^2 = 4^2 + 4^2 + 3^2 + 1^2 = 5^2 + 4^2 + 1^2 = 5^2 + 3^2 + 2^2 + 2^2 \\
43 &= 5^2 + 3^2 + 3^2 = 4^2 + 3^2 + 3^2 + 3^2 = 5^2 + 4^2 + 1^2 + 1^2 \\
44 &= 5^2 + 3^2 + 3^2 + 1^2 = 6^2 + 2^2 + 2^2 \\
45 &= 5^2 + 4^2 + 2^2 = 6^2 + 2^2 + 2^2 + 1^2 = 6^2 + 3^2 = 4^2 + 4^2 + 3^2 + 2^2 \\
46 &= 5^2 + 4^2 + 2^2 + 1^2 = 6^2 + 3^2 + 1^2 \\
47 &= 5^2 + 3^2 + 3^2 + 2^2 = 6^2 + 3^2 + 1^2 + 1^2 \\
48 &= 6^2 + 2^2 + 2^2 + 2^2 = 4^2 + 4^2 + 4^2 \\
49 &= 7^2 = 6^2 + 3^2 + 2^2 = 4^2 + 4^2 + 4^2 + 1^2 = 5^2 + 4^2 + 2^2 + 2^2 \\
50 &= 7^2 + 1^2 = 6^2 + 3^2 + 2^2 + 1^2 = 5^2 + 5^2 = 5^2 + 4^2 + 3^2 = 4^2 + 4^2 + 3^2 + 3^2
\end{aligned}$$

Let the children try to extend the above table up to 100 and let them verify the number of ways for expressing the given number as sum of four or less squares :

For 56, 96	: One way
For 64, 71, 80, 88	: Two ways
For 51, 53, 55, 59, 60, 62, 72, 79, 92, 95	: Three ways
For 57, 61, 63, 65, 67, 68, 69, 77, 78, 83, 87, 94	: Four ways
For 50, 52, 54, 58, 70, 73, 74, 75, 76, 84, 85, 89, 91, 93.	: Five ways
For 66, 81, 97, 99	: Six ways

For 82, 98, 100	: Seven ways
For none	: Eight ways
For 90	: Nine ways

Experiment 119. The Power of Mathematics

The children may be asked to estimate how much time it would take to verify whether a number like 1426753489345 can be expressed as the sum of four or less square numbers and how much time it would take to verify the result for all numbers. The children may be told that when they grow up they would be able to prove the result for all numbers they can think of, in less than one hour.

Experiment 120. Pythagorean Triplets

From the table of square numbers, children may be asked to find three square numbers such that one of these is the sum of the other two. They should be able to find the following Pythagorean triplets :

(3, 4, 5), (6, 8, 10), (9, 12, 15), (12, 16, 20), (15, 20, 25),
 (18, 24, 30), (21, 28, 35), (24, 32, 40), (27, 36, 45), (30, 40, 50),
 (5, 12, 13), (10, 24, 26), (15, 36, 39), (7, 24, 25), (14, 48, 50),
 (8, 15, 17), (16, 30, 34)

They will also find some more triplets. Children may also find three square numbers such that the sum of two is one more or less than the third. Some such triplets are :

(1, 1, 1), (2, 2, 3), (5, 5, 7), (12, 12, 17), (29, 29, 41),
 (70, 70, 99) etc.
 (7, 12, 13), (8, 9, 12), (11, 13, 17), (10, 15, 18), (9, 19, 21),
 (14, 17, 22), (13, 19, 23), (17, 21, 27) etc.

More Patterns in Squares, Cubes and Higher Powers of Natural Numbers

Experiments : 121 to 130

Experiment 121. Finding Truth Set of $\square^2 = 2\triangle^2$ in $\mathbb{N}$

Let the children try to find truth set of

$$\square^2 = 2\triangle^2$$

For this purpose, they may use the table of square numbers they have themselves prepared or they may be given a printed or cyclo-styled table giving squares of all numbers, say from 1 to 100 or 1000. The object is to find a pair of two square numbers such that one square number is double of the other. 1 is a square number, but its double 2 is not; 4 is a square number but its double 8 is not, 9 is a square number but its double 18 is not and so on. The children will find that very often they get a square number very nearly the double of another square number e.g.

$3^2 = 2 \times 2^2 + 1$	$24^2 = 2 \times 17^2 - 2$
$10^2 = 2 \times 7^2 + 2$	$41^2 = 2 \times 29^2 - 1$
$17^2 = 2 \times 12^2 + 1$	$58^2 = 2 \times 41^2 + 2$
	$99^2 = 2 \times 70^2 + 1$

The children will not get a square number which is double of another square number, however far they continue this search. The

children may accept the challenge of finding such a pair of square numbers and for this purpose, they may find squares of larger and larger numbers and get a good deal of well-motivated practice in multiplication. Just as chemists of the middle ages did not succeed in getting a substance which could convert every material into gold, though in this process they learnt a great deal of chemistry; in the same way the children will get a great deal of practice in multiplication, though they will not be able to find the pair of square numbers they were searching for.

Experiment 122. A simple proof that $\square^2 = 2\triangle^2$ has no solution in $\mathbb{N}$

A simple proof can now be given to the children to show that they will never be able to find natural numbers such that the square of one is double that of the other. For this, let them recall the results they have verified earlier viz.

- (i) every natural number is either even or odd
- (ii) the square of even number is even and is a multiple of 4
- (iii) the square of every odd number is odd.

Now there are only four possibilities viz.

- (a) number in $\square$ is odd, number in $\triangle$ is odd
- (b) number in $\square$ is odd, number in $\triangle$ is even
- (c) number in $\square$ is even, number in $\triangle$ is odd
- (d) number in $\square$ is even, number in $\triangle$ is even

We shall rule out each of these possibilities one by one and thus show that it is not possible to find numbers in $\square$ and $\triangle$ so that

$$\square^2 = 2\triangle^2$$

is true

- (a) In this case, L.H.S. is odd and R.H.S. is even. They cannot be equal.
- (b) In this case also L.H.S. is odd and R.H.S. is even. They cannot be equal.
- (c) In this case L.H.S. contains 4 as a factor, while R.H.S. contains only 2 as a factor. They cannot be equal.
- (d) Here number in $\square$ and $\triangle$ are both even. We divide these both by 2. At least one of the numbers may become odd by this process. If this does not happen, we continue to divide by 2, till at least one of the numbers in $\square$ or $\triangle$ becomes odd, so that this case is also reduced to (a) or (b) or (c) and is thus also ruled out.

Thus in each of the four cases L.H.S. cannot be equal to R.H.S., but these are the only four cases. Therefore $\square^2$ can never be equal to $2\triangle^2$. In this way the children can see the power of mathematics. What could not be established in millions of years by trial and error can be established by using logic and mathematics in a matter of minutes and the proof can be understood by students of elementary classes.

Experiment 123. Finding Truth Set of $\square^2=3\triangle^2$

Next let the children try to find the solutions of

$$\square^2 = 3\triangle^2$$

i.e., in the table of squares, they have to find one square number which is three times another square number. They will not succeed, though they will get very near success e.g.

$$7^2 = 3 \times 4^2 + 1$$

$$26^2 = 3 \times 15^2 + 1$$

$$71^2 = 3 \times 41^2 - 2$$

In this case there are nine possibilities and all of these have to be ruled out. If there are any common factors between numbers in $\square$ and $\triangle$, these can be cancelled out so that we can assume that they have no common factors:

- (a) number in $\square$ is of type A and number in $\triangle$ is of type A. This is not possible since in that case these will have a common factor.
- (b) number in $\square$ is of type A and number in $\triangle$ is of type B. This is not possible since L.H.S. would be of type A and would contain 9 as a factor, R.H.S. would also be of type A, but would contain only 3 as a factor.
- (c) number in $\square$ is of type A and number in $\triangle$ is of type C. This is also not possible since L.H.S. would be of type A and would contain 9 as a factor while the R.H.S. would also be of type A but could contain only 3 as a factor.
- (d, e, f) number in $\square$ is of type B and number in $\triangle$ is of type A or B or C. In each of these cases L.H.S. would be of type B and R.H.S. would be of type A and this is not possible.
- (g, h, i) number in $\square$ is of type C and number in $\triangle$ is of type A or B or C. In each of these cases L.H.S. is of type B and R.H.S. would be of type A and this is not possible.

Thus all the nine cases are ruled out and we cannot find natural numbers to satisfy

$$\square^2 = 3\triangle^2.$$

Experiment 124. Finding Truth Set of $\square^2=4\triangle^2$

Next let the children try to find solutions of

$$\square^2 = 4\triangle^2$$

They will be happily surprised to find that here they can find any number of solutions e.g., (2, 1), (4, 2), (6, 3), (8, 4), (10, 5) etc. The number of solutions is infinite.

Experiment 125. Finding Truth Set of $\square^2=5\triangle^2$

Next let the children try to find solution of

$$\square^2 = 5\triangle^2$$

and this time they fail, though they are very near successes e.g.,

$$9^2 = 5 \times 4^2 + 1, 11^2 = 5 \times 5^2 - 4$$

Again the proof of finding the impossibility of finding a solution proceeds in the same way. There are twenty-five cases to be ruled out. However it is easily seen that in $\square^2=5\triangle^2$, the R.H.S. is always of type P and number in $\square$ cannot be of type Q or R or S or T. Thus only five cases remain to be ruled out. Numbers in $\square$ and $\triangle$ cannot be both of type P for we can easily cancel out common factor of 5 or multiples of 5. In other four cases $\square^2$ will contain 25 as a factor, while $5\triangle^2$ would contain only 5 as a factor.

Experiment 126. Finding Truth Set of $\square^2=k\triangle^2$.

In the same way children will find that

$$\square^2 = 6\triangle^2, \square^2 = 7\triangle^2, \square^2 = 8\triangle^2$$

have no solution while $\square^2=9\triangle^2$ has an infinite number of solutions. They will easily be able to guess that $\square^2=k\triangle^2$ has no solution if k is itself not a square number and has an infinite number of solutions if k is a square number.

Experiment 127. Finding Truth set of $\square^3=2\triangle^3$

Again let the children try to find solutions of $\square^3=2\triangle^3$. Here they have to use their own tables of cubes of numbers or they can be given printed or cyclostyled copies of tables of cubes. They again fail to find any solutions and again they can give the proof of the impossibility by examining the four cases. Number in $\square$ cannot be odd, since cube of an odd number is odd and the R.H.S.

is even. If number in $\square$ is even and number in $\triangle$ is odd, the L.H.S. could contain 8 as a factor, while the R.H.S would contain only 2 as a factor. If numbers in $\square$ and $\triangle$ are both even, we can go on dividing by 2 or power of 2 till one of them becomes odd so that this case reduces to one of the three earlier cases.

Experiment 128. Finding Truth Set of $\square^3 = k\triangle^3$

Proceeding in this way, the children will be able to see that $\square^3 = k\triangle^3$ has no solution unless k is a perfect cube and in that case it has an infinity of solutions. Generalising further, they will find that $\square^4 = k\triangle^4$ will have no solution unless k is a perfect fourth power and in that case it has an infinity of solutions and so on and that $\square^n = k\triangle^n$ will have no solutions, unless k is a perfect n th power and in that case it has an infinity of solutions.

Experiment 129. Introducing Square Roots, Cube Roots etc.

Children may be introduced to the idea of square root, cube root, fourth root etc. What is the number whose square is 4? 2 is called the square root of 4.

What is the number whose cube is 27? 3 is called the cube root of 27.

What is the number whose fourth power is 64? 4 is called the fourth root of 64.

What is the number whose square is 2?

The square of 1 is 1 and the square of 2 is 4. There is no natural number whose square is 2. But there may be a fraction whose square is 2. A fraction is ratio of two natural numbers i.e. it may be possible that

$$\frac{\square^2}{\triangle^2} = 2 \quad \text{or} \quad \square^2 = 2\triangle^2$$

but we have already seen that this has no solution in natural numbers. Thus there is no fraction whose square is 2. Similarly there is no natural number or fraction whose square is 3 or 5 or 6 or 7 etc. and there is no natural number or fraction whose cube is 3 or 4 or 5 or 6 or 7.

Experiment 130. Introducing Irrational Numbers

The following numbers are called natural numbers :

1, 2, 3, 4, 5, 6, 7, 8,

The following numbers are called integers :

....., -4, -3, -2, -1, 0, 1, 2, 3, 4, 5,

Any number which is the ratio of two integers with denominator not equal to 0, is called a rational number. We have seen that the square root of 2 is not a rational number. Such a number is called an irrational number. Thus

- (a) the square root of any number which is not a perfect square number is an irrational number.
- (b) the cube root of any number which is not a perfect cube is an irrational number.
- (c) the fourth root of any number which is not a perfect fourth power is an irrational number, and so on.

Experiments : 131 to 135

Experiment 131. Patterns in Quotients

Let the children divide $N=1,000,000,000,000,000,000, \dots$ by 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, $\dots$, and note what they get :

$N \div 2 = 500000000000000000 \dots$
 $N \div 3 = 3333333333333333 \dots$
 $N \div 4 = 250000000000000000 \dots$
 $N \div 5 = 200000000000000000 \dots$
 $N \div 6 = 1666666666666666 \dots$
 $N \div 7 = 1428571428571428 \dots$
 $N \div 8 = 125000000000000000 \dots$
 $N \div 9 = 1111111111111111 \dots$
 $N \div 10 = 100000000000000000 \dots$
 $N \div 11 = 9090909090909090 \dots$
 $N \div 12 = 8333333333333333 \dots$
 $N \div 13 = 7692307692307692 \dots$

Let the children note the patterns :

- (a) When they divide by 2, 4, 5, 8, 10, 16, 20, $\dots$, they begin getting zeros after a certain stage.

- (b) When they divide by other numbers, they find that after a certain stage digits begin to recur. In the case of division by 3 or 9, the same digit occurs again and again. In the case of division by 6 or 12, recurrence begins with the second digit. In other cases a group of digits recur. Thus in division by 7, the group of digits 142857 occurs again and again.

- (c) The number of digits in the recurring group is always less than the divisor.

The children may also notice that the divisors in (a) have only 2 and 5 as factors *i.e.* they are all of the form $2^m 5^n$ where m and n may be 0, 1, 2, 3, $\dots$. They may also note that 2 and 5 are factors of 10 which is the base of our system.

The explanation for (a) is easy to see. The number 1,000,000... has only powers of 2 and powers of 5 as factors. The process of division can therefore start giving zeros after a certain stage only if the divisor is a number with only 2 and 5 or their powers as factors. The process of division also gives an explanation for (c). When the children divide by a given number, the number of different remainders cannot be more than the number itself. Thus when they divide by 13, the only possible remainders are : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. If the remainder at any stage is 0, they begin getting all zeros after that stage. In other cases since there is only a finite number of different possible remainders, the same remainder must occur after a certain stage and as such recurrence must start occurring at that stage. The number of digits in the recurring group thus has to be less than the divisor.

Experiment 132. Patterns in Decimal Quotients.

The children who know decimals can get the patterns as follows :

$1 \div 2 = .5000000000000000 \dots$
 $1 \div 3 = .3333333333333333 \dots$
 $1 \div 4 = .2500000000000000 \dots$
 $1 \div 5 = .2000000000000000 \dots$
 $1 \div 6 = .1666666666666666 \dots$
 $1 \div 7 = .1428571428571428 \dots$
 $1 \div 8 = .1250000000000000 \dots$
 $1 \div 9 = .1111111111111111 \dots$
 $1 \div 10 = .1000000000000000 \dots$
 $1 \div 11 = .0909090909090909 \dots$
 $1 \div 13 = .0769230769230769 \dots$

Two thousand five hundred years ago its approximate value was supposed to be 3, later its approximate value was found correct to four or five places of decimals in India and elsewhere. More than two hundred years ago, a person found its correct value to 700 places of decimals by work extending over a number of years. Now its value can be found correct to thousands of decimals in a matter of minutes by fast electronic computers, but we can never find its exact value even by the fastest electronic computers for even they would require an infinite amount of time.

Experiments : 136 to 145

Experiment 136. Partitions into Two Parts

Take two boxes, say red and blue, and give the child four objects. In how many different ways can he place these four objects in the two boxes? He can place 4 objects in red box and 0 objects in blue box or he can place 3 in red and 1 in blue or 2 in red and 2 in blue or 1 in red and 3 in blue or 0 in red and 4 in blue. There are thus 5 different ways corresponding to

$$4=4+0=3+1=2+2=1+3=0+4$$

Similarly,

$$5=5+0=4+1=3+2=2+3=1+4=0+5$$

$$6=6+0=5+1=4+2=3+3=2+4=1+5=0+6$$

$$7=7+0=6+1=5+2=4+3=3+4=2+5=1+6=0+7$$

and so on. Let the child note the patterns.

(a) The number of ways is always one more than the number of objects.

(b) When the number of objects is seven, the difference in the number of objects in the two boxes is also even and when the number of objects is odd, the difference in the number of objects in the two boxes is also odd.

(c) The child can now conclude that the difference of two numbers is odd or even according as their sum is odd or even. Can he see the reason for this? He easily sees that the sum of two numbers minus difference of these two numbers is always even

$$[(x+y)-(x-y)=2y].$$

He may also note that difference of two even numbers is even and difference of two odd numbers is also even, while the difference of one odd number and one even number is always odd.

$$[2m-2n=2(m-n), (2m+1)-(2n+1)=2(m-n),$$

$$(2m+1)-2n=2(m-n)+1, 2m-(2n+1)=2(m-n-1)+1]$$

Experiment 137. Partitions into three parts.

Next take three boxes, say red, blue and white and give the child 1, 2, 3, 4,.....objects and ask him to place these in the different boxes in all possible ways.

One object : 3 ways

Red	Blue	White	
1	0	0	}
0	1	0	
0	0	1	

Two objects : 6 ways

Red	Blue	White	
2	0	0	}
0	2	0	
0	0	2	
1	1	0	}
1	0	1	
0	1	1	

Three objects : 10 ways

Red	Blue	White	
3	0	0	}
0	3	0	
0	0	3	
2	1	0	}
2	0	1	
1	2	0	
1	0	2	}
0	1	2	
0	2	1	
1	1	1	}

Four objects : 15 ways

Red	Blue	White	
4	0	0	}
0	4	0	
0	0	4	
3	1	0	}
3	0	1	
1	3	0	
1	0	3	}
0	1	3	
0	3	1	
2	2	0	}
2	0	2	
0	2	2	
2	1	1	}
1	2	1	
1	1	2	

(a) Let the children note the systematic method of finding all possible different ways. Without using 0, we can write 4 as 4 or 3+1 or 2+2 or 2+1+1. All the four objects can be put together in one box in three different ways. The four objects can be divided into two groups of 3 and 1 objects and these two groups can be placed in the three boxes in 6 ways. They can also be divided into two groups of 2 each and these two groups can be placed in the boxes in 3 different ways and finally the four objects can be divided into three groups of 2, 1, 1 objects each and these three groups can be placed in the three boxes in 3 different ways.

(b) Let the children complete the following table

Number of objects	1	2	3	4	5	6.....
Number of ways	3	6	10	15	21	28.....

Let the children guess the pattern. The number of ways can be obtained by successively adding 3, 4, 5, 6,..... The children can now continue the table. The children can also make charts showing all the different ways for 3, 4, 5, 6 objects.

Experiment 138. Partitions into four parts.

The children will now be curious to see what pattern emerges if they have four boxes, say red, blue, green and white.

One object : 4 ways

Red	Blue	Green	White
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

Two objects : 10 ways

Red	Blue	Green	White
2	0	0	0
0	2	0	0
0	0	2	0
0	0	0	2
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1

Three objects : 20 ways

Red	Blue	Green	White
3	0	0	0
0	3	0	0
0	0	3	0
0	0	0	3

2	1	0	0
2	0	1	0
2	0	0	1
1	2	0	0
1	0	2	0
1	0	0	2
0	1	2	0
0	1	0	2
0	2	1	0
0	2	0	1
0	0	1	2
0	0	2	1

(a) Let the children now complete the following table :

Number of objects : 1 2 3 4 5.....
Number of ways : 4 10 20 35 56.....

Let the children guess the pattern. What are the successive numbers being added ? These are 6, 10, 15, 21,..... but these are precisely the numbers they got in the last section.

(b) The children may now be able to guess as to what happens when they have five boxes, say red, blue, green, yellow and white and similarly what happens when they have 6 boxes and so on. They get the following table

Number of objects :	1	2	3	4	5	6
Number of ways (with 1 box)	1	1	1	1	1	1
Number of ways (with 2 boxes)	2	3	4	5	6	7
Number of ways (with 3 boxes)	3	6	10	15	21	28
Number of ways (with 4 boxes)	4	10	20	35	56	84
Number of ways (with 5 boxes)	5	15	35	70	126	210

The children may note that every number on this table is the sum of the number on its left and above it. The table can now be continued indefinitely both towards the right and in the downward direction.

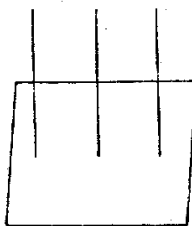
(c) The ability to count and enumerate all possible ways of doing an operation is an important mathematical ability and skill. The child learns to think systematically and with concentration. If he does not proceed systematically or loses concentration, he will miss some possibility and may have to retrace all his steps. Even grown-ups very often miss some possibilities in day-to-day life because of this lack of facility of systematic enumeration.

(d) The child also sees here some patterns emerging and begins to realise that the capacity to recognize patterns can considerably simplify his work. If there were 15 objects and 20 boxes, the direct method of enumeration will take days and even then one would not be sure whether one is right. With the help of the above table prepared by recognizing patterns, one can get the answer in a matter of minutes.

(e) It may also be emphasized that recognizing a pattern does not ensure that it will always hold, though it makes it highly likely to be true. To establish the pattern completely we need some more advanced mathematics and the children may be motivated to learn more mathematics in order to work out the proofs.

Experiment 139. Partitions on an Abacus.

What we did in Experiment 136 has an interesting interpretation on an abacus. Suppose there are three needles (unit's needle, ten's needle and hundred's needle) and we are given 2 beads. How many different numbers can we show on the abacus? Obviously these are the 6 numbers 200, 020, 002, 110, 101, 011. Arguing in the same way, with 3, 4, 5, 6, 7, 8 and 9 beads we can show respectively 10, 15, 21, 28, 36, 45, and 55 numbers. How many numbers can we show with 10 beads? The next number in the sequence is 66, but it is obvious that the partitions 10, 0, 0; 0, 10, 0; 0, 0, 10 are not possible since no needle can have more than 9 beads. Thus the total number of numbers we can show with 10 beads is 63. The next number in the sequence is 78, but with 11 beads, the following partitions are not possible: 11, 0, 0; 0, 11, 0; 0, 0, 11; 10, 1, 0; 1, 10, 0; 0, 10, 1; 0, 1, 10; 10, 0, 1; 1, 0, 10. So the total number of numbers which can be shown with 11 beads is 69. The total number of partitions for 12 is 91, but the following partitions are not permissible:



12, 0, 0; 0, 12, 0; 0, 0, 12; 11, 1, 0; 1, 11, 0; 0, 11, 1;
0, 1, 11; 11, 0, 1; 1, 0, 11; 10, 1, 1; 1, 10, 1; 1, 1, 10.

The total number of different numbers which can be shown is thus 79. The children may now continue similar arguments for 13, 14, 15 beads. They may also investigate the results when

there are 2 needles or 4 needles in the abacus. They may also investigate the results when each needle is required to have at least one bead so that we have to find the total number of partitions of a given number into a given number of non-zero addends when none of the addends can exceed 9.

Experiment 140. Partitions into Non-Zero Addends.

Without using zero, let the children try to express a given number as a single number, as a sum of two addends, as a sum of three addends and so on and find the total number of ways in which this can be done. Thus we have

Number 2 : Two ways

2 1+1

(1) (1)

Number 3 : Four ways

3 2+1 1+1+1
1+2

(1) (2) (1)

Number 4 : Eight ways

4 3+1 2+1+1
2+2 1+2+1 1+1+1+1
1+3 1+1+2

(1) (3) (3) (1)

Number 5 : Sixteen ways

5 4+1 3+1+1 2+1+1+1
1+4 1+3+1 1+2+1+1 1+1+1+1+1
3+2 1+1+3 1+1+2+1
2+3 2+2+1 1+1+1+2
2+1+2
1+2+2

(1) (4) (6) (4) (1)

Number 6 : Thirty two ways

6 5+1 4+1+1 3+1+1+1 2+1+1+1+1
 4+2 1+4+1 1+3+1+1 1+2+1+1+1 1+1+1+1+1+1
 3+3 1+1+4 1+1+3+1 1+1+2+1+1
 2+4 3+2+1 1+1+1+3 1+1+1+2+1
 1+5 3+1+2 2+2+1+1 1+1+1+1+2
 1+2+3 2+1+2+1
 1+3+2 2+1+1+2
 2+1+3 1+1+2+2
 2+3+1 1+2+2+1
 2+2+2 1+2+1+2

(1) (5) (10) (10) (5) (1)

The number 6 can be expressed as the sum of one addend in one way, as a sum of two addends in 5 ways, as a sum of three addends in 10 ways, as a sum of four addends in 10 ways, as a sum of five addends in 5 ways and as a sum of six addends in one way. The children can continue the process with the numbers 7 and 8.

Experiment 141. Number of Partitions into Non-Zero Addends.

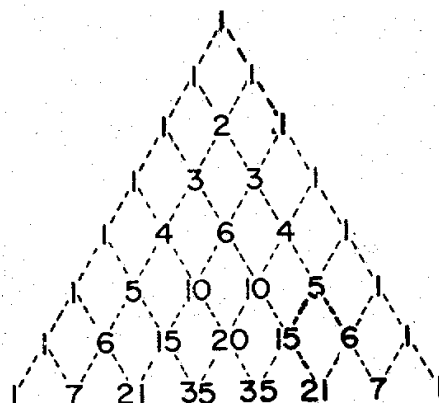
The children can now prepare the following table :

Number	1	2	3	4	5	6	7	8
Total number of ways	1	2	4	8	16	32	64	128

The pattern is now obvious. With the help of this pattern, they see that the total number of partitions of 16 would be 32,768. Without noticing the pattern and trying by direct enumeration, it will take a long time and even then the answer may not be correct, if some mistake is made in the process.

Experiment 142. Pascal Triangle Pattern.

Another pattern which the children notice is the following :



This pattern is known as the **Pascal Triangle** and will occur again and again in our discussions. Here each number is obtained by adding the two numbers just above it, one on its left and the other on the right. Noticing this pattern, we can continue the table as long as we like. We can now interpret the last line as follows :

The number 8 can be expressed as the sum of one addend in one way, as the sum of two addends in seven ways, as the sum of three addends in twenty-one ways, as the sum of four addends in thirty-five ways, as the sum of five addends also in thirty-five ways, as the sum of six addends in twenty-one ways, as the sum of seven addends in seven ways and as the sum of eight addends in one way.

To find the number of ways in which a given number can be broken up into another given number of addends, we look up the line in the above table corresponding to the first given number and then look up the number on this line corresponding to the second given number. It will be useful for the children to read the table in this way and then actually verify some results by direct enumeration.

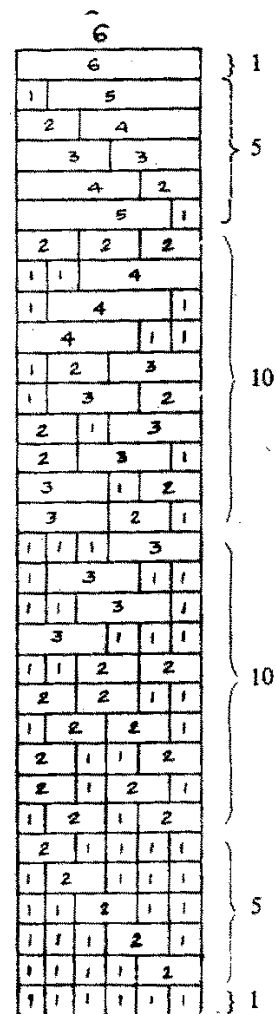
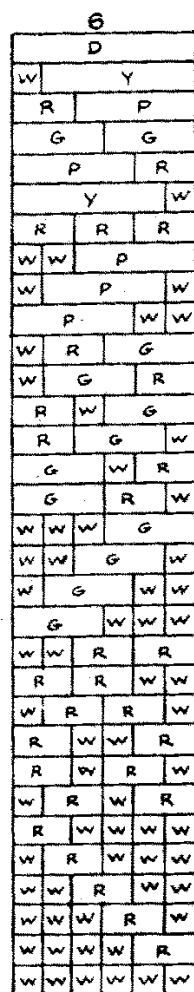
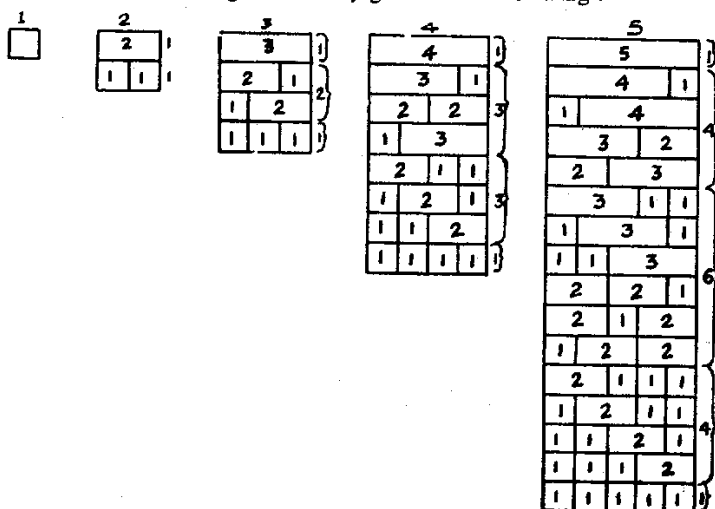
The children can also note the symmetry in the above table from the right and the left.

Experiment 143. Partition with Cuisenaire Rods.

A useful way of representing these partitions is by using Cuisenaire rods. These are rectangular rods of the following lengths and colours :

Length	Colour	Symbol
1	White	W
2	Red	R
3	Green	G
4	Purple	P
5	Yellow	Y
6	Dark green	D
7	black	K
8	brown	N
9	blue	U
10	Orange	O

Any number of such sets are made available to the children. A child is given a rod (say of length 5). He has to find first one rod of length 5, then get two rods with total length 5, then get three rods of total length 5, then get four rods of total length 5 and finally he has to get five rods of total length 5 and he has to do in all possible ways. Different children can be given rods of different lengths for this purpose. The arrangements they get are the following :



Experiment 144. Pascal Triangle Pattern in Powers of Eleven.

Let the children note the patterns in the following multiplications :

$$\begin{aligned} 1 \times 11 &= 11 \\ 11 \times 11 &= 121 \\ 11 \times 11 \times 11 &= 1331 \\ 11 \times 11 \times 11 \times 11 &= 14641 \end{aligned}$$

$$\begin{array}{r} 1 \ 4 \ 6 \ 4 \ 1 \\ 1 \ 4 \ 6 \ 4 \ 1 \\ 1 \ 4 \ 6 \ 4 \ 1 \\ 1 \ 5 \ (10) \ (10) \ 5 \ 1 \\ 1 \ 5 \ (10) \ (10) \ 5 \ 1 \\ 1 \ 6 \ (15) \ (20) \ (15) \ 6 \ 1 \end{array}$$

These give the patterns of the Pascal Triangle.

Experiment 145. Figurate Numbers.

Let the numbers in the Pascal Triangle be written according to the inclined lines. Then we get the arithmetic triangle in the simplest form.

1	1	1	1	1	1	1	1	
1	2	3	4	5	6	7	8	
1	3	6	10	15	21	28	36	
1	4	10	20	35	56	84	120	
1	5	15	35	70	126	210	330	
1	6	21	56	126	252	462	792	
1	7	28	84	210	462	924	1716	
1	8	36	120	330	792	1716	3432	
1	9	45	165	495	1287	3003	6435	

To find any number we add the number just above it and just on its left. Alternatively to find any number in this triangle, we find the sum of all the numbers in the row above the row of this number and upto the number just above this number or find the sum of all

the numbers in the column on its left and upto the number just to the left of this number. As illustrations we have :

$$\begin{aligned} 84 &= 1+3+6+10+15+21+28=1+6+21+56 \\ 70 &= 1+4+10+20+35=1+4+10+20+35 \\ 36 &= 1+7+28=1+2+3+4+5+6+7+8 \end{aligned}$$

The numbers in successive horizontal rows (or in successive vertical columns) are called **figurate number** of 1st, 2nd, 3rd order etc.

The figurate number of first order are all odd.

The figurate number of second order are alternatively odd and even.

The figurate number of third order are : 2 odd, 2 even, 2 odd, 2 even.

The figurate number of fourth order are : 1 odd, 3 even, 1 odd, 3 even.

The figurate number of fifth order are : 4 odd, 4 even, 4 odd, 4 even,.....

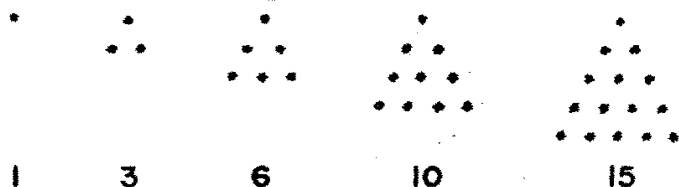
Let the children continue this pattern.

Number Patterns in Geometrical Shapes

Experiments: 146 to 155

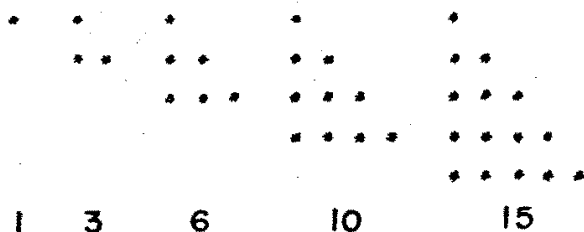
Experiment 146. Triangular Numbers.

Let the children arrange dots as follows :

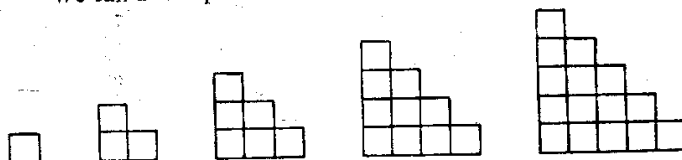


The children can continue the pattern. The numbers they get this way viz., 1, 3, 6, 10, 15, 21,.....are called triangular numbers and are called so for obvious reasons.

The dots can also be arranged as follows :



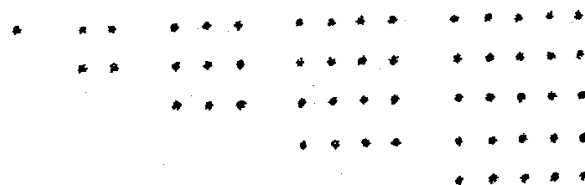
We can also represent these on graph paper as follows :



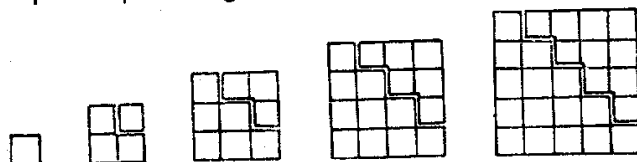
From the shape of these, we can also call these *stair-case numbers*.

Experiment 147. Square Numbers.

Let the children combine two consecutive triangular numbers or stair-case numbers and see what they get :

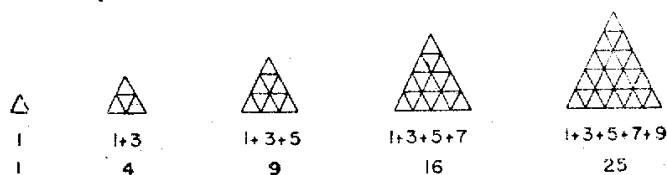


1 4 9 16 25



These numbers are called *square numbers* and again for obvious reasons. The children can now easily deduce that the sum of two consecutive triangular numbers is a square number.

The square number can also be represented as follows :

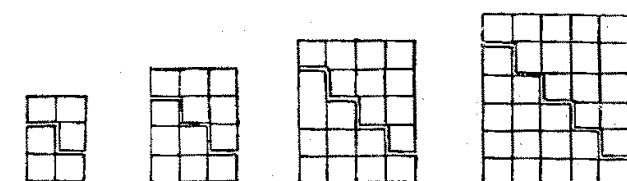


The children can now deduce that the sum of any number of consecutive odd numbers is a square number which is the square of

the number of odd numbers. They can now be asked to find the sum of the first 20 odd numbers or the sum of the first 100 odd numbers.

Experiment 148. The Sum of First N Natural Numbers.

Let the children see what they get by adding a staircase number to itself.



$$\begin{array}{llll} 2 \times 3 & 2 \times 6 & 2 \times 10 & 2 \times 15 \\ = 2 \times 3 & = 3 \times 4 & = 4 \times 5 & = 5 \times 6 \end{array}$$

We find

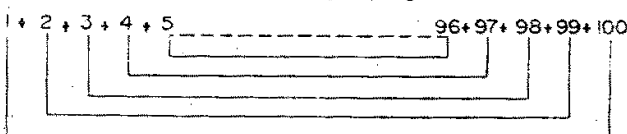
$$\begin{array}{ll} 2 \times (1+2) & = 2 \times 3 \\ 2 \times (1+2+3) & = 3 \times 4 \\ 2 \times (1+2+3+4) & = 4 \times 5 \\ 2 \times (1+2+3+4+5) & = 5 \times 6 \end{array}$$

The children can easily see the pattern viz. :

$2 \times (1+2+3+4+\dots \text{ up to any number}) = (\text{that number}) \times (\text{that number} + 1)$ so that the sum of any number of natural numbers is equal to half the product of this number with the next consecutive natural number e.g.,

$$1+2+3+\dots+100 = (100 \times 101) \div 2 = 5050$$

The same result can be easily seen by writing the numbers as follows :



and then combining the first with the last, the second with the last but one and so on ; it is easily seen that in this way we get 50 pairs of numbers and the sum of numbers of each pair is 101, so that the sum of the first hundred numbers is $50 \times 101 = 5050$.

If we have to add the first ninety-nine numbers, we get 49 pairs with sum 100 each and one isolated number viz., 50 so that the sum of first ninety-nine numbers is 4950.

The recognition of pattern now enables us to find the sum of any number of natural numbers e.g., the sum of the first 10,000 natural numbers is $(10,000 \times 10,001) \div 2$. Without this pattern, it would have taken many hours to find this sum.

The formulae we have found can be written as

$$1+2+3+4+\dots \text{ to } n \text{ terms} = n \times (n+1) \div 2$$

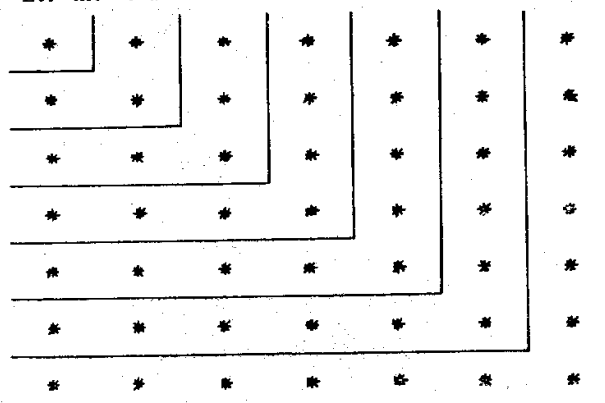
$$1+3+5+7+\dots \text{ to } n \text{ terms} = n^2$$

$$2+4+6+8+\dots \text{ to } n \text{ terms} = 2(1+2+3+\dots \text{ to } n \text{ terms})$$

$$= \frac{2(n+1)n}{2} = n(n+1).$$

Experiment 149. Finding Square Roots.

Let the children note the following geometrical pattern



and deduce

$$\begin{array}{rcl} & & 1 = 1^2 \\ & & 1 + 3 = 2^2 \\ & & 1 + 3 + 5 = 3^2 \\ & & 1 + 3 + 5 + 7 = 4^2 \\ & & 1 + 3 + 5 + 7 + 9 = 5^2 \\ & & 1 + 3 + 5 + 7 + 9 + 11 = 6^2 \\ & & 1 + 3 + 5 + 7 + 9 + 11 + 13 = 7^2 \\ & & \dots \dots \dots \end{array}$$

This geometrical pattern again verifies that the sum of the first n odd natural numbers is n^2 .

Conversely any square number can be decomposed into the sum of a number of odd natural numbers. If a given number can be expressed as the sum of first n natural odd numbers, then the number must be the square of the number n , which will accordingly be the square root of the given number. This fact can be used to find the square root of small numbers.

To find the square root of any number, we subtract from it consecutively 1, 3, 5, 7, 9, 11,..... The number of times we have to subtract gives the square root of the given number. Thus to find the square root of 81 and 110, we get

81	110
<u>1</u>	<u>1</u>
80	109
<u>3</u>	<u>3</u>
77	106
<u>5</u>	<u>5</u>
72	101
<u>7</u>	<u>7</u>
65	94
<u>9</u>	<u>9</u>
56	85
<u>11</u>	<u>11</u>
45	74
<u>13</u>	<u>13</u>
32	61
<u>15</u>	<u>15</u>
17	46
<u>17</u>	<u>17</u>
0	29
	19
	10

From 81, we have to subtract the first 9 odd natural numbers, to get 0. Therefore square root of 81 is 9. From 110, we have to subtract first 10 odd natural numbers and we cannot subtract the first 11 odd natural numbers. Thus 110 is not a square number and its square root lies between 10 and 11.

Experiment 150. Finding Cube Roots and Fourth Roots.

The same method can be used to find the cube roots, fourth roots etc. of small numbers. If we form geometrical patterns in 3 dimensional space, say with small wooden cubes and partition them in the same way as before we get,

$$\begin{aligned} 1 &= 1^3 \\ 1 + 7 &= 2^3 \\ 1 + 7 + 19 &= 3^3 \\ 1 + 7 + 19 + 37 &= 4^3 \\ 1 + 7 + 19 + 37 + 61 &= 5^3 \end{aligned}$$

Thus to find the cube root of a number, we have to subtract consecutively 1, 7, 19, 37, 61, 91,..... These numbers are easily seen to be $1^3, 2^3 - 1^3, 3^3 - 2^3, 4^3 - 3^3, 5^3 - 4^3, \dots$

Similarly to find the fourth root of a number, we have to subtract consecutively $1^4, 2^4 - 1^4, 3^4 - 2^4, 4^4 - 3^4, 5^4 - 4^4, \dots$

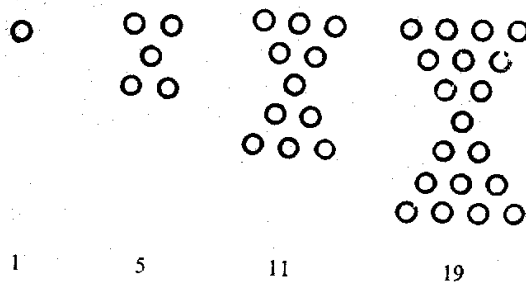
The children may find the first ten numbers to be subtracted for fourth, fifth and sixth roots.

This would give them incidentally some well-motivated practice in subtraction and multiplication.

It may also be noted that though the geometrical pattern breaks down for fourth and fifth roots etc., the mathematical pattern still continues to operate.

Experiment 151. Number Patterns in Other Shapes.

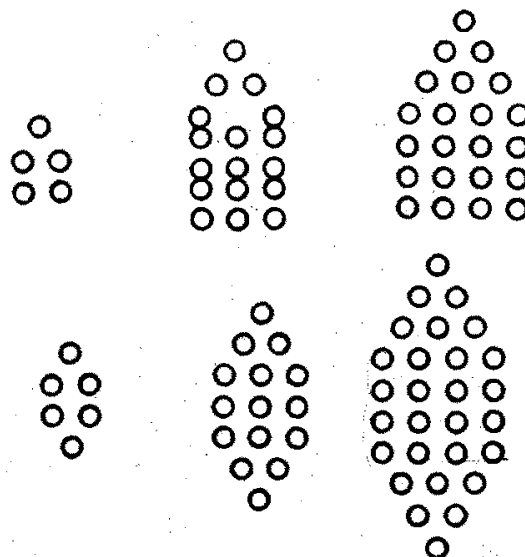
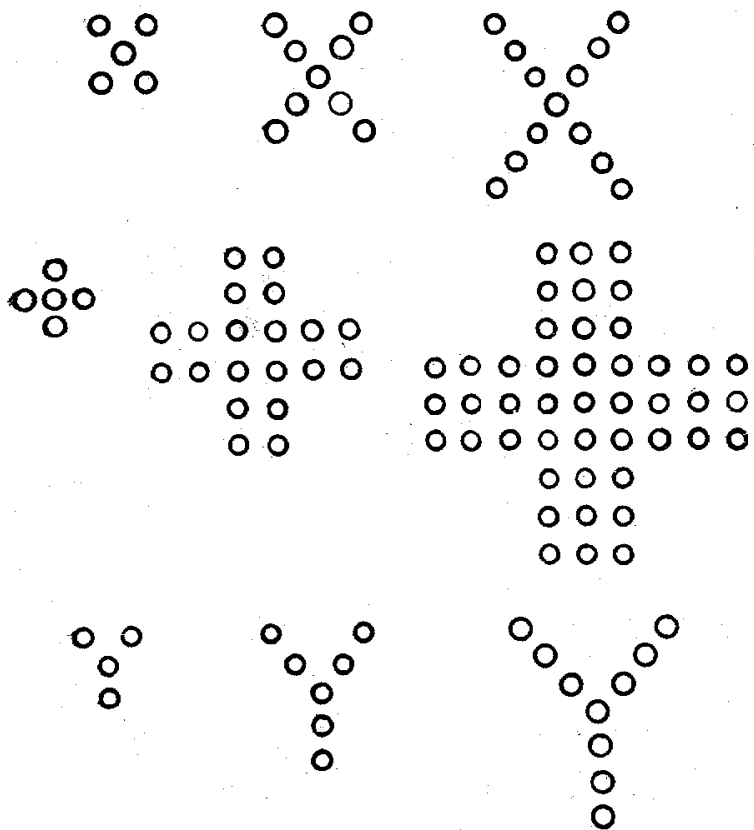
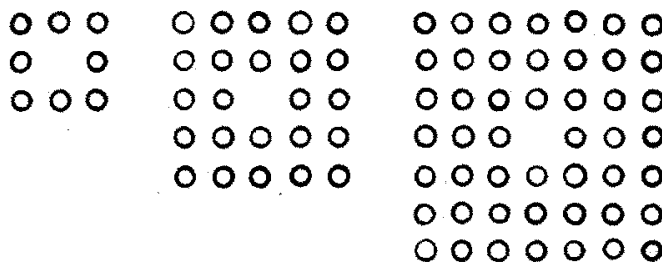
We have found the number patterns corresponding to triangles and squares. The children can form their own geometrical patterns and then find the corresponding number pattern e.g.,



These may be called double triangular numbers. The children may try to find the tenth or fifteenth double triangular numbers. The following observation would be helpful :

- (a) The successive differences are 2×2 , 2×3 , 2×4 , 2×5 ,
- (b) Each double triangular number can be obtained by subtracting one from double of the corresponding triangular number.

Similarly the children may find 10th numbers of the following patterns



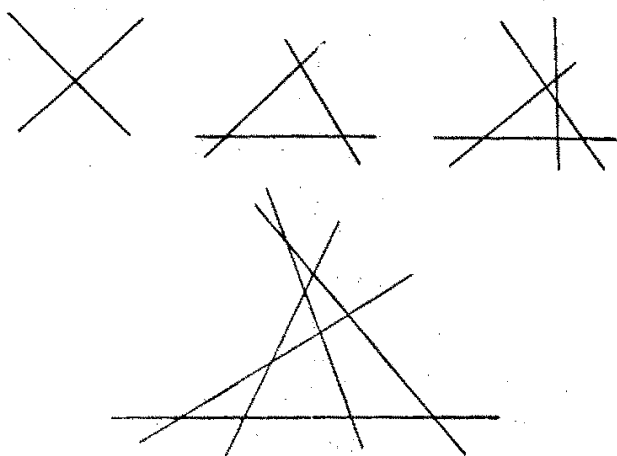
In most cases, successive differences will form arithmetic progressions.

Experiment 152. Number Patterns in Three-Dimensional Shapes.

Children may be given square plates of different sizes and may be asked to form triangular pyramids of marbles or of cubical pieces of wood and note the patterns of numbers obtained.

Experiment 153. Number Pattern in Intersections of Straight Lines.

Two straight lines can intersect in at most one point, three straight lines can intersect in at most three points, four straight lines can intersect in at most six points, five straight lines can intersect in at most ten points and so on. The number of points of



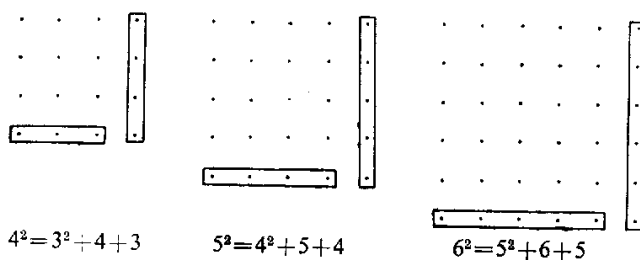
intersection are successively 1, 3, 6, 10, which are triangular numbers. If we draw a sixth line in the plane, it can intersect the other five lines in at most five points so that the maximum number of points of intersections will now be 15. With a seventh line, six more points of intersection can be added to get a maximum 21 points of intersection. It will be an interesting experience for the children to draw lines with the maximum number of points of intersection and see the triangular number pattern. In this case they also get a proof of the pattern. They may also find the pattern in the number of distinct non-overlapping triangles formed or in the number of regions into which the plane has been divided in each case. They may also find the patterns in the maximum number of lines in which 2, 3, 4, 5, 6, planes can intersect and in the maximum number of regions into which the space is divided by these planes.

Experiment 154. Algebraic Formulae Through Geometrical Patterns.

Certain algebraic formulae can be deduced from a careful study

of geometrical patterns. We consider some examples below. Children can discover more patterns themselves.

(a)



$$4^2 = 3^2 + 4 + 3$$

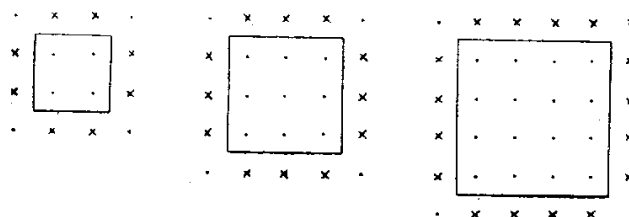
$$5^2 = 4^2 + 5 + 4$$

$$6^2 = 5^2 + 6 + 5$$

Children can deduce :

$$n^2 = (n-1)^2 + n + (n-1)$$

(b)



$$4^2 = 2^2 + 4 \times 2 + 4$$

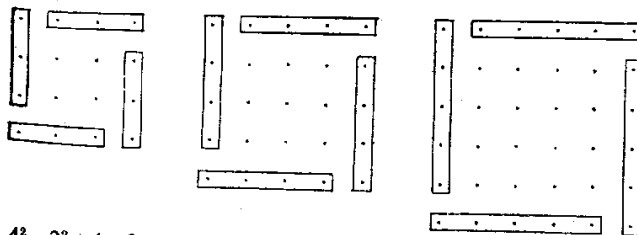
$$5^2 = 3^2 + 4 \times 3 + 4$$

$$6^2 = 4^2 + 4 \times 4 + 4$$

The children deduce :

$$n^2 = (n-2)^2 + 4(n-2) + 4$$

(c)



$$4^2 = 2^2 + 4 \times 3$$

$$5^2 = 3^2 + 4 \times 4$$

$$6^2 = 4^2 + 4 \times 5$$

The children deduce :

$$n^2 = (n-2)^2 + 4(n-1)$$

(d) By working with three dimensional analogues of the above models (beads on wires etc.), children can deduce the following formulae :

$$\begin{aligned}n^3 &= (n-1)^3 + n^2 + (n-1)^2 + n(n-1) \\n^3 &= (n-2)^3 + 6(n-2)^2 + 12(n-2) + 8 \\n^3 &= (n-2)^3 + 2n^2 + 2n(n-2) + 2(n-2)^2\end{aligned}$$

In the first case, they start with a cube of n^3 beads and remove one corner cube of $(n-1)^3$ beads. They are left with 3 faces. They take one face completely giving n^2 beads. They then take the $n(n-1)$ beads remaining in the second face and $(n-1)^2$ beads remaining in the third face.

In the second case, they start with a cube of n^3 beads and remove the central cube of $(n-2)^3$ beads. They are left with 6 faces. From each face they take the central square of $(n-2)^2$ beads. They are left with 12 edges. From each edge they take the central $(n-2)$ beads and then they are left with the 8 corner beads.

In the third case they again start with a cube of n^3 beads and remove the central cube of $(n-2)^3$ beads. They are again left with 6 faces. They take 2 opposite faces with n^2 beads, then two other opposite faces with removing $n(n-1)$ beads each and finally the remaining 2 faces with $(n-1)^2$ beads each.

It is obvious that the children can obtain a large variety of patterns themselves. This will again strengthen their geometrical intuition and the capacity to arrange things systematically.

Experiment 155. Another Pattern for Triangular and Square Numbers.

Let the children verify that

$$\begin{aligned}1^2 + 2^2 + 2^2 &= 3^2 \\2^2 + 3^2 + 6^2 &= 7^2 \\3^2 + 4^2 + 12^2 &= 13^2 \\4^2 + 5^2 + 20^2 &= 21^2 \\5^2 + 6^2 + 30^2 &= 31^2\end{aligned}$$

What pattern do they see ?

They take the sum of squares of two consecutive numbers and add to it the square of their product. They get the square of a number which is greater than the product by unity i.e.

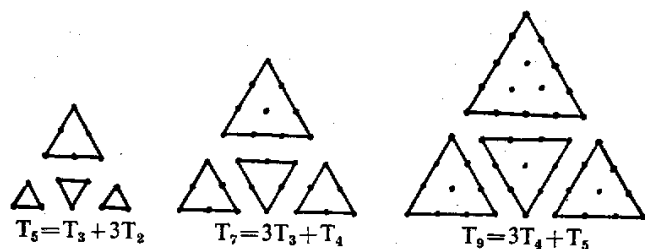
$$n^2 + (n+1)^2 + \{n(n+1)\}^2 = \{n(n+1)+1\}^2$$

The children can verify it for any value of n

Now the n th triangular number T_n is $\frac{n(n+1)}{2}$. So we get

$$n^2 + (n+1)^2 + (2T_n)^2 = (2T_n + 1)^2$$

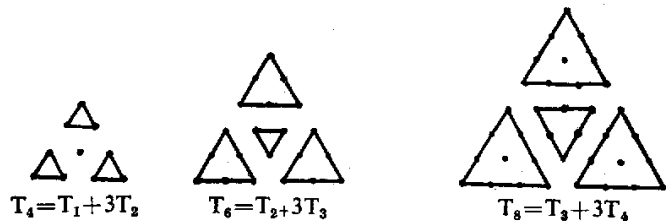
Again let them break bigger triangular pattern into smaller triangular patterns :



What pattern do they notice ? They find

$$3T_n + T_{n+1} = T_{2n+1}$$

Let us now take triangular number of even order and break them into smaller triangular patterns.



The general pattern is

$$T_{2n} = T_{n-1} + 3T_n$$

Patterns with Polyminoes

Experiments : 156 to 165

Experiment 156. Patterns in Biminoes.

Here the children can work with graph paper ruled with squares. They may be asked to colour two adjacent squares *i.e.* two squares with one side in common. The patterns that emerge are



but these are not essentially different patterns, for if we cut one and put it on the other, it exactly fits the other. Thus with two squares, we get essentially one pattern. We say that there is only ONE BIMINO ('bi' means 'two') or Domino.

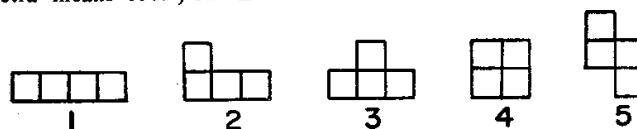
Experiment 157. Patterns in Triminoes.

Next let the children form patterns with three squares. After some trial and error, the children find only two different TRIMINOES ('tri' means 'three') or TROMINOES *viz.*



Experiment 158. Patterns in Tetraminoes.

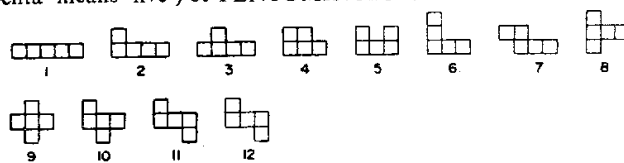
With four squares, there are five different TETRAMINOES ('tetra' means 'four') or TETROMINOES *viz.*



In the first, all four squares are in one row ; in the second, three squares are in one row and the fourth is on one side adjacent with one of the corner squares ; in the third, three squares are in one row and the fourth is on one side adjacent with the middle square ; in the fourth, two squares are in one row and the other two are on one side adjacent with each of these squares and finally in the fifth, two squares are in one row and the other two are on either side adjacent with each of these two squares. It will be seen that this exhausts all the possibilities and any pattern the children will get will be congruent with one or the other of these five, as can be seen by cutting and fitting.

Experiment 159. Patterns in Pentaminoes.

With five squares there are TWELVE different PENTAMINOES ('penta' means 'five') or PENTOMINOES *viz.*

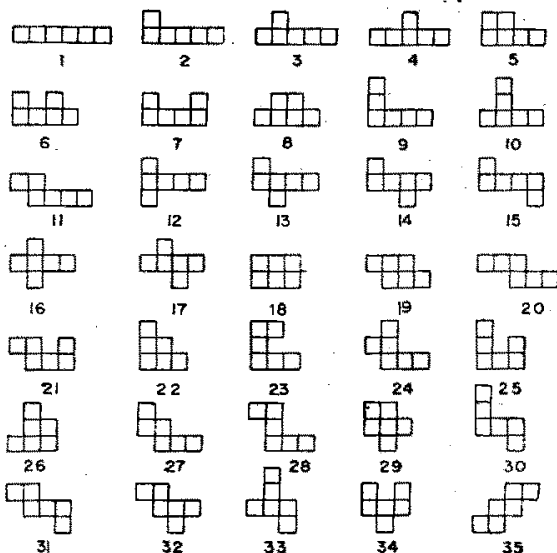


In the first pattern, all five squares are in one row ; in the next two patterns four squares are in one row and the fifth is adjacent with one of the corner squares or one of the middle squares ; in the next four patterns, three squares are in one row and the other two are on one side either adjacent with one corner square and one middle square or adjacent with two corner squares or with one square adjacent with a corner square and the other one adjacent with this ; in the next four patterns, three squares are in one row and the other two are on opposite sides either of the same corner square or of the same middle square or of one corner and of one middle square or of different corner squares and finally in the last pattern, two squares are in one row, two are on one side and one is on the other side.

The children may find these patterns by trial and error, but they should be led to think of the systematic way given away or some similar version of it.

Experiment 160. Patterns in Hexaminoes.

With six squares, there are THIRTY-FIVE possible HEXAMINOES ('hexa' means 'six') or HEXOMINOES viz.



These may be classified as follows :

All six squares in one row :	No. 1
Five squares in one row and one on one side :	Nos. 2-4
Four squares in one row and two on one side :	Nos. 5-11
Four squares in one row and two on opposite sides :	Nos. 12-17
Three squares in one row and three on one side in one row :	Nos. 18-21
Three squares in one row and three on one side in two rows :	Nos. 22-28

Three squares in one row and three on two opposite sides :

Nos. 29-34

Two squares in three rows :

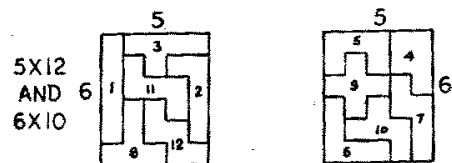
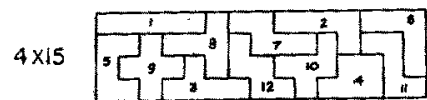
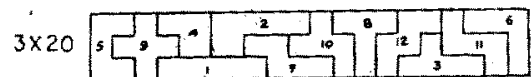
No. 35

A similar type of classification will help in writing all possible HEPTAMINOES ('hepta' means 'seven'), or HEPTOMINOES and OCTAMINOES ('octa' means 'eight') or OCTOMINOES but it will require a great deal of concentration to draw all these patterns. There are 108 distinct heptominoes including one with a 'hole' shown below :



Experiment 161. Making Rectangles with Polyminoes.

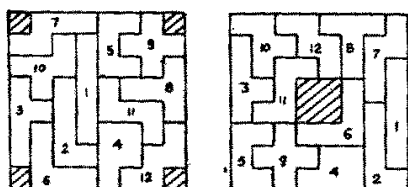
The children may like to make hard paper or flannel or card board or wooden models of the various polyminoes and many activities can be built round these models e.g. 12 pentaminoes cover 60 squares. The question can be raised whether with these 60 squares, we can make rectangles of sizes 3×20 , 4×15 , 5×12 and 6×10 . The answer is 'yes' in each case and the patterns are given below :



The children may colour these different pentaminoes with different colours and get beautiful designs.

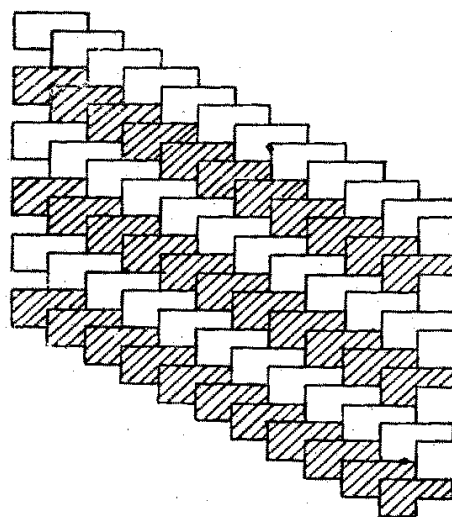
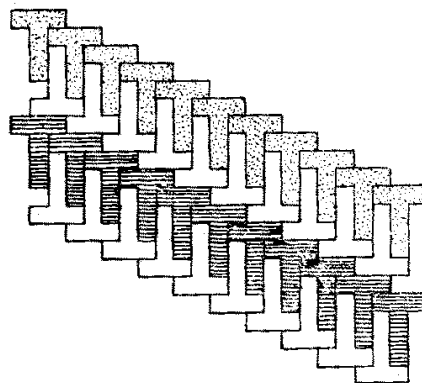
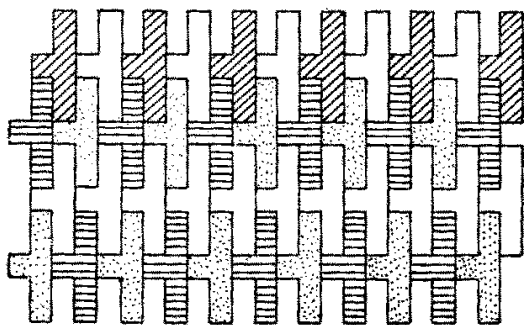
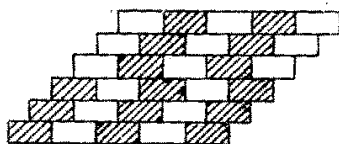
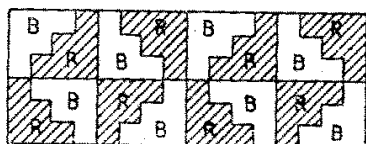
Experiment 162. Squares with Pentaminoes.

The children can also try to get a square of size 8×8 from the twelve pentaminoes; of course four spaces will remain vacant. These can be arranged at the corner or at the centre. Two solutions are given below. Children may find others.



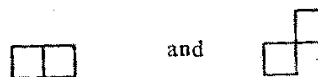
Experiment 163. Tessellations with Polyminoes.

The children will find it an interesting exercise to try to cover the floor with a single polymino used as a tile and to find for which polyminoes this is possible and for which it is not. By using different colours, beautiful designs can be obtained e.g. we give some designs with hexaminoes.



Experiment 164. Modified Polyminoes.

In making polyminoes, we had considered only squares which touch along a side. Suppose we allow squares which are allowed to 'touch' at one corner also so that both



are permissible. With three squares one can have



Let the children now find all possible generalised polyminoes with four, five and six squares.

Experiment 165. Odd and even Hexaminoes.

Let the children shade the thirty-five hexaminoes so that if one square is shaded, the adjacent square (or squares) are unshaded. In 24 hexaminoes, 3 are shaded, and 3 are unshaded, these are called odd hexaminoes. In the remaining hexaminoes, the numbers shaded and unshaded are both even; these are called even hexaminoes. Let the children find which are even and which are odd.

Similarly let them classify tetraminoes and pentaminoes into odd and even ones.

Experiments : 166 to 175

Experiment 166. Recurrence Relation for Partitions.

We have already come across the Pascal triangle pattern viz.

			1			
		1	1			
	1	2	1			
	1	3	3	1		
	1	4	6	4	1	
	1	5	10	10	5	1
	1	6	15	20	15	6

We shall denote by $(6,4)$ the fourth number of the sixth row so that

$$(6,4)=10, (7,3)=15, (7,5)=15, (5,2)=4, (8,3)=21$$

The children may find similarly $(9,4)$, $(10,5)$, $(11,7)$, $(12,9)$.

Each of these above equations has an interpretation in terms of partitions. This $(7,5)=15$ means that there are 15 ways in which the number 7 can be expressed as a sum of 5 non-zero addends.

The children can find relations between these numbers e.g.

$$(7,5)=(6,4)+(6,5); (6,4)=(5,3)+(5,4); (5,4)=(4,3)+(4,4).$$

The children can now generalise these to

$$(n, r) = (n-1, r-1) + (n-1, r)$$

By writing explicitly all the partitions on both sides and establishing correspondences between these, the children can get an idea of the proof of this general relation e.g. for the second equation above :

$$\begin{aligned} (5,3) & \left\{ \begin{array}{l} (1, 1, 3) \rightarrow (1, 1, 1, 3) \\ (1, 3, 1) \rightarrow (1, 1, 3, 1) \\ (3, 1, 1) \rightarrow (1, 3, 1, 1) \\ (1, 2, 2) \rightarrow (1, 1, 2, 2) \\ (2, 1, 2) \rightarrow (1, 2, 1, 2) \\ (2, 2, 1) \rightarrow (1, 2, 2, 1) \end{array} \right\} \quad (6,4) \\ (5,4) & \left\{ \begin{array}{l} (1, 1, 1, 2) \rightarrow (2, 1, 1, 2) \\ (1, 1, 2, 1) \rightarrow (2, 1, 2, 1) \\ (1, 2, 1, 1) \rightarrow (2, 2, 1, 1) \\ (2, 1, 1, 1) \rightarrow (3, 1, 1, 1) \end{array} \right\} \end{aligned}$$

The partitions of n into r addends can be divided into the following two categories :

- those which begin with 1. These can be obtained by just adding addend 1 to each partition of $(n-1)$ into $r-1$ parts.
- those which begin with a number more than 1 i.e. with 2 or 3 etc. These can be obtained by adding 1 to the first addend of each partition of $(n-1)$ into r parts.

Let the children verify this rule for a number of examples. This will also make clear the structure of the Pascal triangle.

Experiment 167. Partitions and Pascal Triangle.

The children may also note the similarity between the pattern in the tables of experiments of Chapter Five where we find the number of ways in which a given number of objects can be placed in a given number of boxes, and the pattern of the Pascal triangle.

Experiment 168. Pascal Triangle Pattern in Powers of Eleven.

We also come across the same Pascal triangle pattern by considering

(11) , $(11)^2$, $(11)^3$, $(11)^4$ etc. Thus

$$(n-1, 1) \quad (n-1, 2) \dots (n-1, n-1)$$

$$\begin{array}{c} (n-1, 1) \quad (n-1, 2) \dots (n-1, r) \dots (n-1, n-1) \\ (n-1, 1) \quad (n-1, 2) \quad (n-1, 3) \dots (n-1, r-1) \dots (n-1, n-1) \\ (n-1, 1) \dots ((n-1, r) + (n-1, r)), \dots (n-1, n-1) \end{array}$$

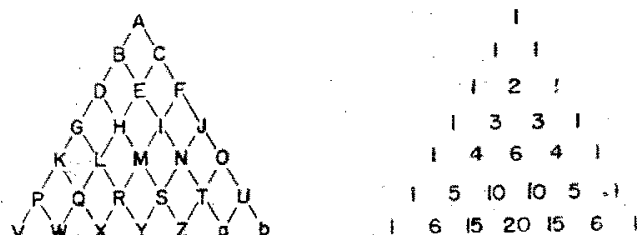
So that $(n, r) = (n-1, r) + (n-1, r-1)$

The children can verify this by doing a number of particular cases. It may also be seen that this formula will also be true in other scales. In the scale of 7, the Pascal triangle will be as follows :

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 13 13 5 1
1 6 21 26 21 6 1
1 10 30 50 50 30 10 1

Experiment 169. Pascal Triangle Pattern in Zigzags.

Let us consider the zig-zig shown below.



A particle starts from the point A. What is the number of possible paths for it to reach any node of this zig zig? Let us consider the problem for it to reach nodes in fourth row.

node	paths.	number of paths
G	ABDG	1
H	ABDH, ABEH, ACEH	3
I	ABEI, ACEI, AFI	3
J	ACFJ	1

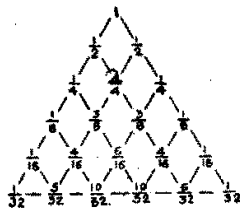
The number of paths are again determined by the Pascal triangle.

The reason for this is obvious. To reach M, the particle must pass through H or I. The number of paths to M is thus the sum of the number of paths to H and I. Thus if (n, r) now denotes the number of paths for reaching r th point in n th row, then

$$(n, r) = (n-1, r-1) + (n-1, r)$$

Experiment 170. Pascal Triangle and Probability.

As particle proceeding from A, will pass through one and only one node of each row. Thus it can pass through either K or L or M or N or O. There are 16 paths which start for A and terminate in this row. Out of these only 1 passes through K, 4 pass through L, 6 pass through M, 4 pass through N and only 1 passes through O. If we consider a large number of paths and all paths are equally likely, then only one-sixteenth of these pass through K, four-sixteenths pass through L, six-sixteenths pass through M, four-sixteenths pass through N and one-sixteenth pass through O. Similarly out of 32 paths starting from A and terminating in the next row, the number of paths terminating at P, Q, R, S, T and U are 1, 5, 10, 10, 5, 1 respectively and the respective proportions are $1/32$, $5/32$, $10/32$, $10/32$, $5/32$, and $1/32$. These ratios are called probabilities of passing through these points. The children can write all the probabilities as below :



Experiment 171. Pascal Triangle Pattern in Coin Throwing.

If a child throws one coin, there are two possibilities viz. Head or Tail

H T
1 1

If a child throws two coins, there are four possibilities viz.

H H H T T T
1 2 1

If a child throws three coins, there are eight possibilities viz.

H H H H H T H T T T
1 3 3 1

(all three heads) (2 heads & 1 tail) (1 head & 2 tails) (all three tails)

If a child throws four coins, there are sixteen possibilities viz.

H H H H	H H H T	H H T T	T T T H	T T T T
	H H T H	H T H T	T T H T	
	H T H H	H T T H	T H T T	
	T H H H	T H T H	H T T T	
		T T H H		
		T H H T		
1	4	6	4	1
(all four heads)	(3 heads and 1 tail)	(2 heads and 2 tails)	(1 head and 3 tails)	(all four tails)

The Pascal triangle pattern is easily seen to emerge. Children can continue this pattern one or two stages more.

The relative proportions for the throw of four coins are : $1/16$, $4/16$, $6/16$, $4/16$, $1/16$. We say the probability of getting all four heads is $1/16$, of getting 3 heads and 1 tail is $4/16$, of getting 2 heads and 2 tails is $6/16$, of getting 1 head and 3 tails is $4/16$ and of getting all tails is $1/16$.

Experiment 172. Pascal Triangle Pattern in Binary Decisions.

A person may be asked a question and he may reply 'Yes' (Y) or 'No' (N). For two persons the possibilities are two yeses, one yes and one no and two nos. The pattern that thus emerges is again the Pascal Triangle pattern.

One person	Y	N		
	1	1		
Two persons	Y Y	Y N, N Y	NN	
	1	2	1	
Three persons	Y Y Y	Y Y N, Y N Y, N Y Y	Y N N, N Y N, N N Y	NNN
	1	3	3	1
Four persons	Y Y Y Y	Y Y Y N	Y Y N N	Y N N N
		Y Y N Y	Y N Y N	N Y N N
		Y N Y Y	N Y N Y	N N Y Y
		N Y Y Y	N N Y Y	N N N Y
			N Y N Y	
			N Y Y N	
	1	4	6	4
				1

Experiment 173. Pascal Triangle Pattern in Human Relations.

A person has two parents viz. one father (F) and one mother (M). He has four grand-parents viz. 1 Father's Father (FF), 2 Father's

Mother (FM) and Mother's Father (MF) and 1 Mother's Mother (MM). Proceeding in this way, children get again the Pascal triangle pattern.

Parents	F	M			
	1	1			
Grand parents	FF	FM, MF	MM		
	1	2	1		
Great grand parents	FFF	FFM, FMF, MFF	FMM, MFM, MMF	MMM	
	1	3	3	1	
Great great grand parents.	FFFF	FFFM	FFMM	FMMM	MMMM
		FFMF	FMFM	MFMM	
		FMFF	FMMF	MMFM	
		MFFF	MMFF	MMMF	
			MFFM		
			MFMF		
	1	4	6	4	1

The interpretation of the last line is that among the 16 great great grand parents, one is related to the individual by throughout male ancestors, four are related through three male and one female, ancestors, six are related through 2 male and 2 female ancestors, four are related through one male and three female ancestors and one is related through female ancestors only.

Experiment 174. Pascal Triangle Pattern in Subsets of A Set.

Let us find the number of subsets of a given set. If a set consists of 4 objects, it will have 1 subset of four objects, 4 subsets of three objects each, 6 subsets of 2 objects each, 4 subsets of one object each and 1 subset of no object (i.e. empty set)

	{ABC}	{AB}	{A}	
	{ABD}	{AC}	{B}	
	{ACD}	{AD}	{C}	{ }
{ABCD}	{BCD}	{BC}	{D}	
		{BD}		
		{CD}		
1	4	6	4	1

Thus the children get the following table :

Number of objects in the set	Number of subsets containing					
	0 object	1 object	2 objects	3 objects	4 objects	5 objects
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

We again get a Pascal triangle pattern.

Let (n, r) denote the number of subsets containing r objects from a set containing n objects, then the children can show that

$$(n, r) = (n-1, r-1) + (n-1, r)$$

The first term in the right gives the number of subsets including a particular object and the other gives the number of subsets excluding this particular object.

If we write I for 'included' and E for 'excluded', we get the following table :

One object set	I	E		
	1	1		
Two objects set	II	IE, EI	EE	
	1	2	1	
Three objects set	III	IIE, IEI, EII	IEE, EIE, EEI	EEE
	1	3	3	1
Four objects set	IIII	IIIE IIIEI IEIE EIII	IIIE IEIE IEEI EEII EIEI EIEE	IEEE EIEE EEIE EEEE
	1	4	6	4
				1

Here IEEI means that the first and fourth objects are included and second and third are excluded.

Experiment 175. Pascal Triangle Pattern as an Illustration of Isomorphism and Abstraction in Mathematics.

The children have seen that the same Pascal triangle pattern emerges in so many different situations viz. partitions, multiplications, paths in a zig-zig, probability, throwing of coins, yes-no opinions, heridity trees and subsets of a set and this list is far from exhaustive. Whenever there are two alternatives at each stage, this pattern can emerge.

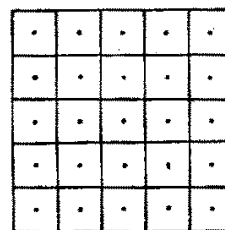
Moreover some of the tables above differ in symbols only e.g. they differ only in having HT or FM or IE. If we interchange the symbols, the tables become the same.

Though the situations are different, symbols are different, yet the mathematical pattern is the same. This is responsible for the power of mathematics. The same mathematical pattern may be applicable to a larger number of entirely different situations. Mathematicians study an abstract pattern and then apply the results to all the different situations. Though the situations are different, the pattern is the same. We say the situations are **isomorphic** (iso=same, morphic=form). This recognition of a common pattern in different situations is known as **abstraction** and this capacity of abstraction is responsible for the great power of mathematics.

Experiments: 176 to 195

Experiment 176. Construction of a Geoboard.

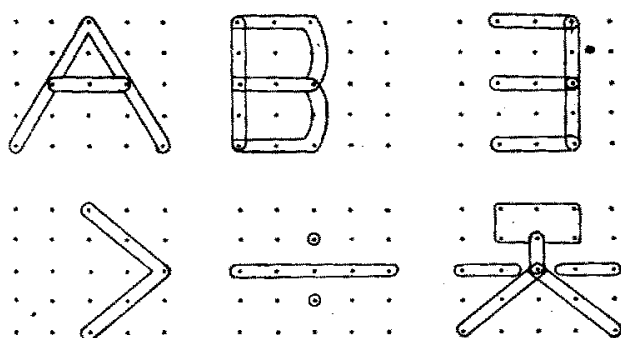
A geoboard may be made for each child on a $3/8$ " plywood of size $12" \times 12"$ divided into 25 equal squares, with one pin centrally placed in each square.



Smaller 9 pin and 16 pin geoboards can also be useful. Each child is given packets of rubber bands of different colours. Geometrical figures can be made on the board with the help of rubber bands and many interesting geometrical properties can be studied easily. The figures can be removed as easily as they are made. Moreover the child can give free play to his creativity and imagination.

Experiment 177. Freeplay Activity on the Geoboard.

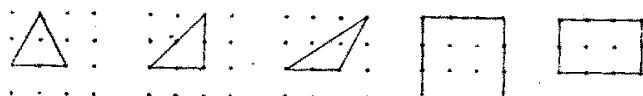
The simplest activity can be for the child to write all the letters of the alphabet, all the digits, all the mathematical symbols and other figures with rubber bands e.g.



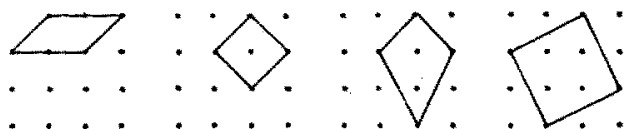
The teacher can draw some patterns on the board and the children can try to copy these on the geoboard or the children can make their own patterns and interpret them.

Experiment 178. Geometrical Patterns on Geoboards.

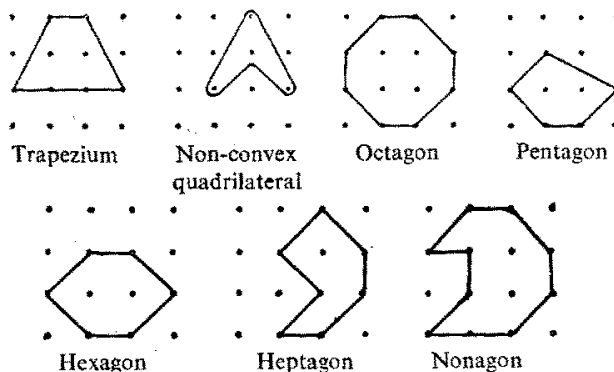
Next let the children make geometrical patterns viz. triangle, quadrilateral, pentagon etc. and they may study their classification.



Isosceles triangle Right angled triangle Obtuse angled triangle Square Rectangle



Parallelogram Rhombus Kite Convex quadrilateral



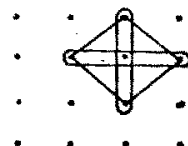
Experiment 179. Demonstration of Geometrical Results on Geoboards.

Children can easily prove some 'theorems' on the geoboard e.g.

(a) A diagonal bisects a parallelogram

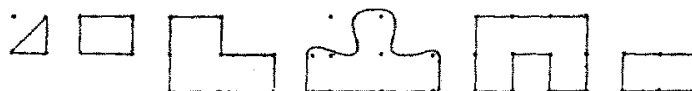


(b) Diagonals of a rhombus bisect each other at right angles



Experiment 180. Areas of Figures with no Pins Inside.

Children may find areas of figures formed by these figures which contain pins on the boundary only and no pins strictly inside.



Let the children draw more figures with pins on the boundary and note the pattern.

No. of pins	3	4	6	8	10	12
Area	$\frac{1}{2}$	1	2	3	4	5

Can they see any pattern? Yes, the area is one less than half the number of pins or

$$A = \frac{1}{2}p - 1,$$

where A is area in terms of squares and p is the number of pins.

Experiment 181. Areas of figures with Pins Inside.

Children may now try to find a formula for the areas of these polygons which have pins both inside and on the boundary



No. of pins on the boundary	p :	8	12	7	12
No. of pins inside	q :	1	3	2	4
Area	A :	4	8	4½	9

Now $A = \frac{1}{2}p - 1$ does not hold. Let us find the discrepancy and how it is related to q. In every case this is equal to q so that

$$A = \frac{1}{2}p + q - 1$$

The children now can proceed to verify this formula and they would be pleasantly surprised to find it always holds.

Experiment 182. Deformation to Increase or Decrease Area Included by a Given Amount.

Given a certain figure on the board, the children may be asked to deform it so as to increase or decrease the area by a given amount. Each additional pin on the boundary will increase the area by half a unit and each additional pin in the interior will increase the area by a unit.



Deformation 1 increases the area by 1

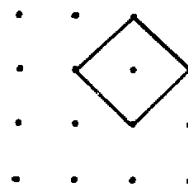
Deformation 2 increases the area by 2

Deformation 3 decreases the area by 1

Deformation 4 decreases the area by 2

Experiment 183. Counting the Number of Geometrical Figures Possible.

Children may be asked to find how many triangles, right-angled triangles, squares, rectangles, parallelograms etc. they can draw in a 9 pin or 16 pin geoboard. It may be desirable to hang all the classroom geoboards on the wall and ask each child to make a different square from others and so on. Squares of the type shown below should not be forgotten.

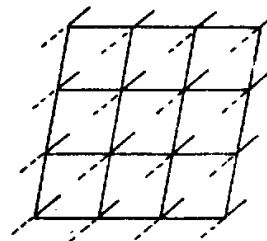


Experiment 184. Polyminoes on Geoboards.

Children may make all the biminoes, triminoes, tetraminoes, pentaminoes and hexaminoes on their geoboards. Thirty-five geoboards may be used to show all the thirty-five hexaminoes. The boundary of each polyminoe may be shown by a single band, otherwise a separate band may be used to show each square.

Experiment 185. A Geomesh.

A modified geoboard may be made with a wire mesh (this may be called a geomesh) with nails or pins or thin rods in both directions at each node



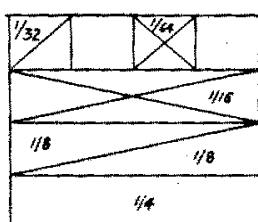
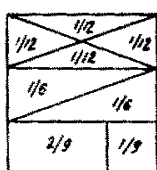
The advantage is that it can not only be rotated around like a geoboard, but it can be turned about and as such the congruency of polyminoes can be more easily seen here.

Experiment 186. Number of Different Polygons on Geoboards.

Children may like to investigate the largest number of different figures which can be formed with 4 pins, 5 pins, 6 pins only.

Experiment 187. Fractions on Geoboards.

Children may study fractions with the help of a geoboard.



$$\frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$$

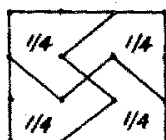
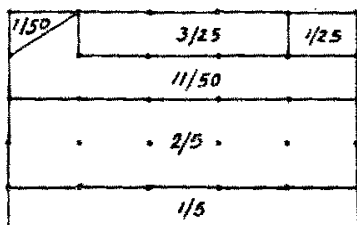
$$\frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

$$\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$$\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4}$$

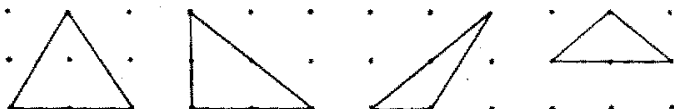
$$\frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3}$$



Experiment 188. Finding Formulae for Areas of Geometrical Figures.

Children can discover the formulae for the area of various figures with the help of a geoboard and Pickes formulae ($A = \frac{1}{2} p + q - 1$) e.g.

(a) they can draw a number of triangles



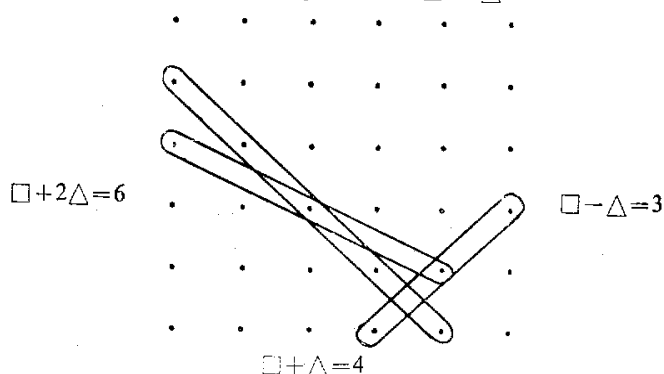
(b) they can draw parallelograms of different sizes and find that area = base $\times$ height

(c) They can draw trapezia of different sizes and discover that area = $\frac{1}{2}$ (sum of opposite sides) $\times$ height

Experiment 189. Showing Solution Sets of Equations on Geoboards.

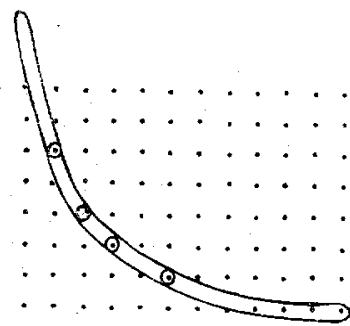
The children can use the geoboard to show solution sets of sentences like

$$\square + \triangle = 4, \quad \square - \triangle = 3, \quad \square + 2\triangle = 6$$



with larger geoboards, they can also show solution sets of sentences like

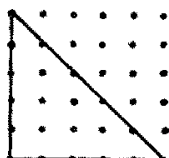
$$xy = 12$$



Experiment 190. Showing Solution sets of Inequalities.

The children can also show the solution sets of inequalities here e.g. solution set of $x + y < 5$, $x > 0$, $y > 0$

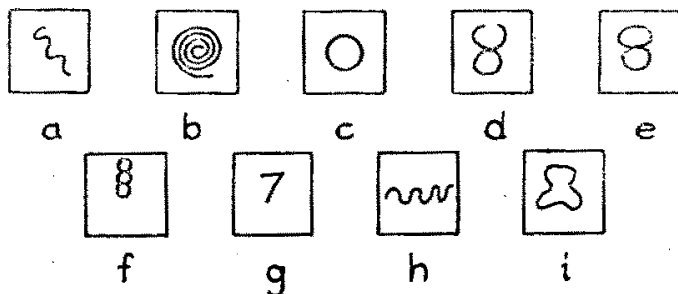
is shown by pins inside the triangle below



Experiment 191. Non-Metrical Geometry on Geoboards.

The children may be asked to place their geoboards flat on their desks so that the side with no pins protruding is up.

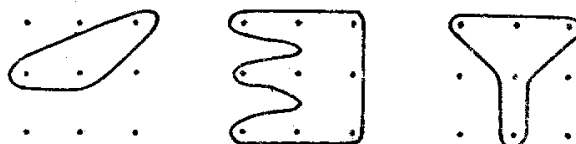
Give two or three strings to each child and ask him to make figures on the flat surface of the board.



Explain to the children the difference between open curves (a, b, d, g, h) and closed curves (c, e, f, i). There are curves here which have crossover points (d, e, f). A closed curve with no crossover points is called a simple closed curve (e, i). The curve h represents a wave.

Experiment 192. Number of closed curves with given characteristics.

The children may be asked to turn their geoboards so that the pin side is up. With their strings which may be knotted to form closed loops, they may be asked to form simple closed curves with no pin inside, one pin inside, two pins inside, three pins inside, nine pins inside, four on the boundary and one inside and so on.



How many such simple closed curves can they draw? They have to partition the number 9 into three addends (number of pins inside, number of pins on and number of pins outside the string). The figures above correspond to the partitions $0+0+9$, $1+0+8$, $2+0+7$, $3+0+6$, $9+0+0$, $4+1+4$.

For solving this problem, we have to find the number of solutions of

$$\square + \triangle + \diamond = 9$$

in non-negative integers. It is obvious that if the numbers in $\square$, $\triangle$ are fixed, the number in $\diamond$ is determined uniquely. Moreover numbers in $\square$, $\triangle$ satisfy the sentence

$$\square + \triangle \leq 9$$

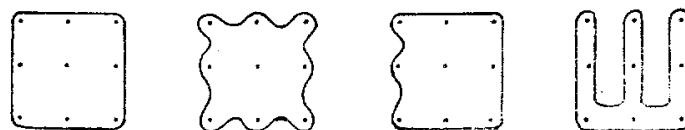
The solution set of this may be found by the children as the union of the solution sets of

$$\begin{aligned} \square + \triangle = 0, \square + \triangle = 1, \square + \triangle = 2, \square + \triangle = 3, \square + \triangle = 4, \\ \square + \triangle = 5, \square + \triangle = 6, \square + \triangle = 7, \square + \triangle = 8, \square + \triangle = 9. \end{aligned}$$

The number of solutions of these equations are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, so that the total number of solutions is

$$1+2+3+4+5+6+7+8+9+10=55$$

Even when we fix particular pins inside, on and outside the curve, the string can be deformed horizontally to take an infinite number of different shapes e.g. with nine pins inside, we can have



All these figures will be called topologically equivalent. They are not same but they can be continuously deformed into one another without lifting the string from the board and without violating the pins conditions.

Even when the number of pins is fixed, we can have different non-equivalent figures depending on which pins are inside, which are on the boundary and which are outside. Thus if there is to be one pin inside, no pin on the boundary and eight pins outside, the number of different non-equivalent figures is nine. Children can now try to enumerate all topologically different figures.

The total number of different figures with no pins on the boundary is 512 and the total number of non-equivalent figures is 155102.

For bright children, the concept of combinations can be introduced. One pin may be chosen in 9 ways, a pair of pins in 36 ways, a triplet of pins in 84 ways and so on. The total number of non-equivalent figures with no pins on the boundary is thus

$$9c_0 + 9c_1 + 9c_2 + 9c_3 + \dots + 9c_9 = 2^9 = 512.$$

If we allow pins on the boundary, the calculation is slightly more complicated. If x pins are in the interior and y are on the boundary, the total number of figures is $9c_x \times (9-x)c_y$, so that the total number of non-equivalent figures is given by

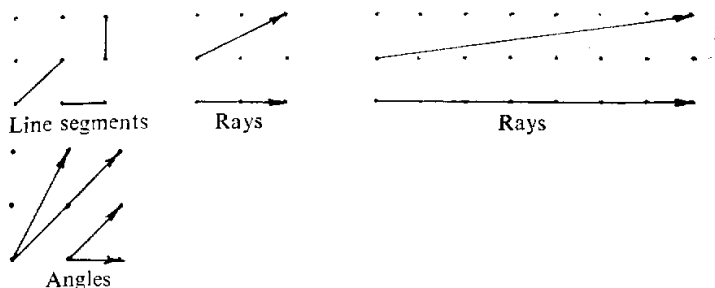
$$\begin{aligned} &9c_0 (9c_0 + 9c_1 + 9c_2 + \dots + 9c_9) \\ &+ 9c_1 (9c_0 + 9c_1 + \dots + 9c_8) \\ &+ 9c_2 (9c_0 + 9c_1 + \dots + 9c_7) \\ &+ 9c_3 (9c_0 + 9c_1 + \dots + 9c_6) \\ &+ 9c_4 (9c_0 + 9c_1 + \dots + 9c_5) \\ &+ 9c_5 (9c_0 + 9c_1 + \dots + 9c_4) \\ &+ 9c_6 (9c_0 + 9c_1 + \dots + 9c_3) \\ &+ 9c_7 (9c_0 + 9c_1 + 9c_2) \\ &+ 9c_8 (9c_0 + 9c_1) \\ &+ 9c_9 (9c_0) \\ &= 155102 \end{aligned}$$

The ambitious children may now try to find the total number of non-equivalent figures for a 16 pin geoboard and even for a 25 pin geoboard.

If we allow rotations of the board, then with one pin inside, there are three distinct figures corresponding to the pin being a central pin or a corner pin or a pin in the middle of an edge. Similarly the children will find it interesting to find how many distinct figures are possible with two pins, three pins inside etc.

Experiment 193. Line segments, rays and angles.

Children can draw line segments, rays and angles with the help of strings on geoboards.



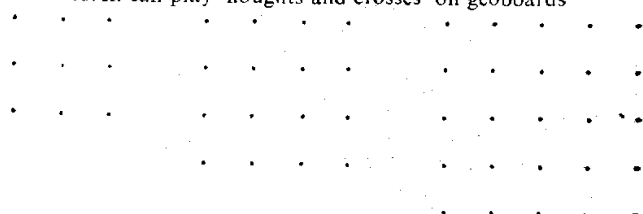
A line segment is finite in length. A ray has a starting point in the board, but it goes outside the board. To give the children the idea of a ray extending to infinity, a number of geoboards may be combined together and yet the ray will go beyond. Even if they combine all the geoboards of the school, the ray will go beyond and it will go beyond even if they combine all the geoboards in the whole world. Children can draw two rays starting at the same point. They will form an angle. Children may form angle with no pin in the interior, 2 pins in the interior and even all the 9 pins in the interior



Children may like to investigate as to how many angles have no pins inside, one pin inside or all the 9 pins inside.

Experiment 194. Noughts and Crosses on Geoboards.

Children can play 'noughts and crosses' on geoboards



A pair of children can use rubber bands of different colours. The player who can get three bands of his colour in one line (row, column

or diagonal) first wins the game. Similarly on 16 pin board or 25 pin board the child who succeeds first in getting 4 or 5 rubber bands of his colour in a line will win the game. A modified version of the game is when bands of both colours are available to both players and a player can use either colour in any move. The player who completes the first line in one colour wins the game. Another version with 25 pins board can be when child can place two bands simultaneously and these may be of one colour or two colours.

Experiment 195. Abstract Version of Noughts and Crosses.

Child can also play an abstract version of this game

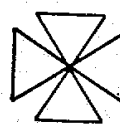
(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

Each pin is associated with a number pair. The first number gives the number of the row and the second number gives the number of the column. Now these number pairs are written on cardboard discs and these are placed on the table in a random order. Two children pick up the discs alternately. The child who first gets 5 discs with the same first number or the same second number or with the same sum equal to 6 or with the same difference equal to 0 wins the game (why?)

Experiments: 196 to 205

Experiment 196. Forming Figures with Equilateral Triangles.

Each child may be given an equilateral triangle and a cardboard piece or a hard board paper piece and asked to cut triangles equal to the given triangle and then to arrange these to get different shapes. The children may get shapes like the following :



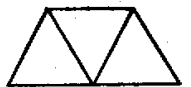
Flower



Star



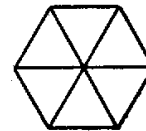
Rhombus



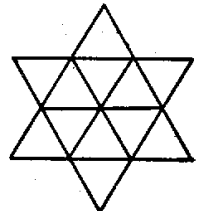
Trapezium



Parallelogram



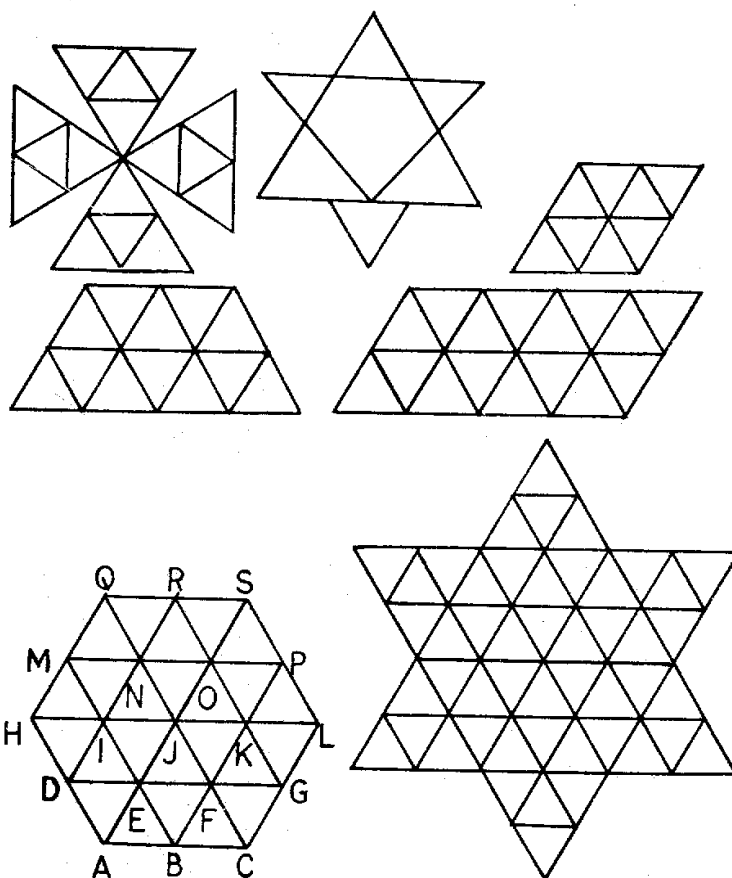
Hexagon



Star

Experiment 197. Enlargement of Figures.

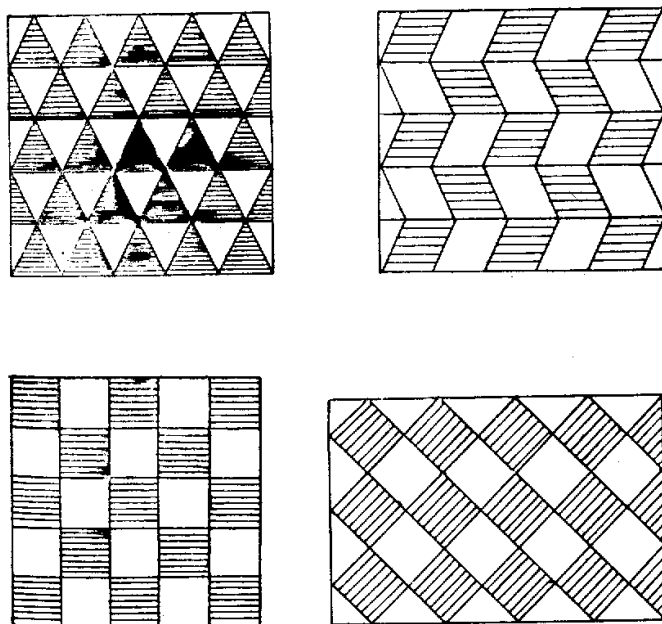
Let the children add triangles to get figures of double the sizes they got earlier.



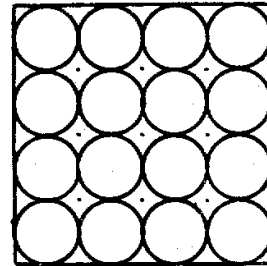
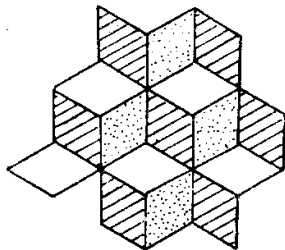
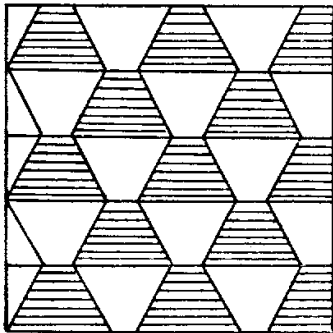
recover from the bigger figures. They can distinguish between the over-lapping and non-overlapping cases e.g. in the case of the trapezium they can get four non-overlapping trapezia ABHF, BCGH, CDEG, and EFHG. From the hexagon or the star, they can get only one non-lapping original shape, but if overlapping shapes are allowed, they can get quite a larger number e.g. hexagons ABFJID, BCGKJE, GLPOJF, PSRNJK, RGMIOJ, HBEJNM, EFKONI are all hexagons with the same size as original. The centres of these hexagons are E, F, K, O, N, F and J.

Experiment 198. Tessellations.

The children may now 'tessellate' the top surfaces of their desks or portions of the floor of their classrooms with equilateral triangles, parallelograms, rhombuses, squares, hexagons, etc. as follows :



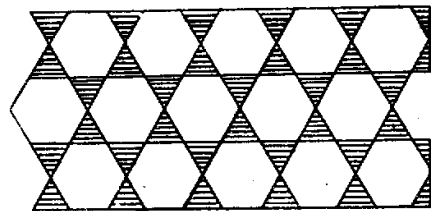
In each case to get double the size, we require four times the number of triangles. How many triangles do we require to get three times the size or four times the size? The children may also reduce the new figures to half size by removing triangles. They may also see how many of the original shapes in original sizes they can



The children have to wait for the proof of the result that they will succeed in these three cases and not in others.

Experiment 200. Tessellations with Combinations of Polygons.

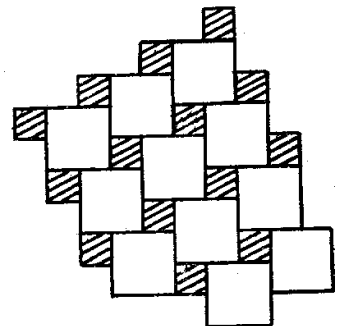
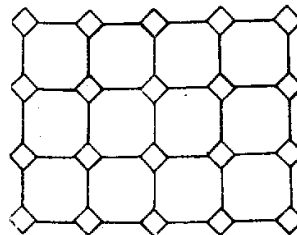
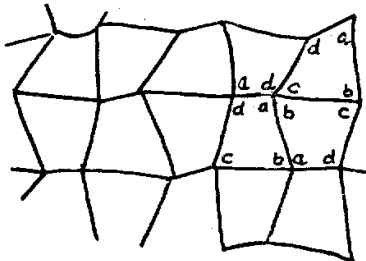
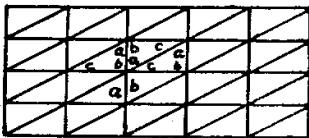
Though children cannot tessellate with an octagon alone, they may be able to do so by combining it with some other figures. Similarly beautiful designs can be obtained by combining different polygons *e.g.*

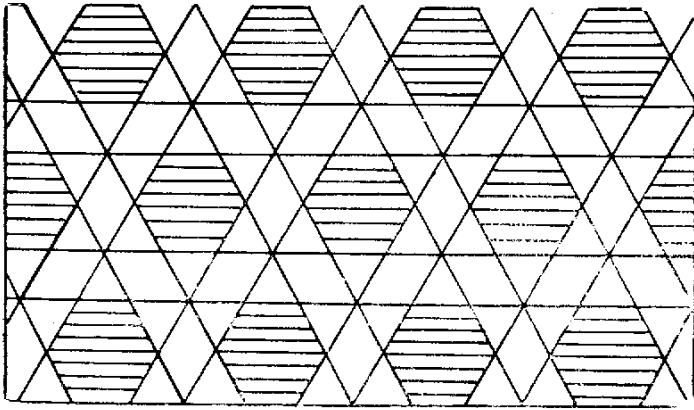


Except for the edges, the floor or desk surface can be completely covered and beautiful designs can be obtained.

Experiment 199. Polygons which can Tessellate.

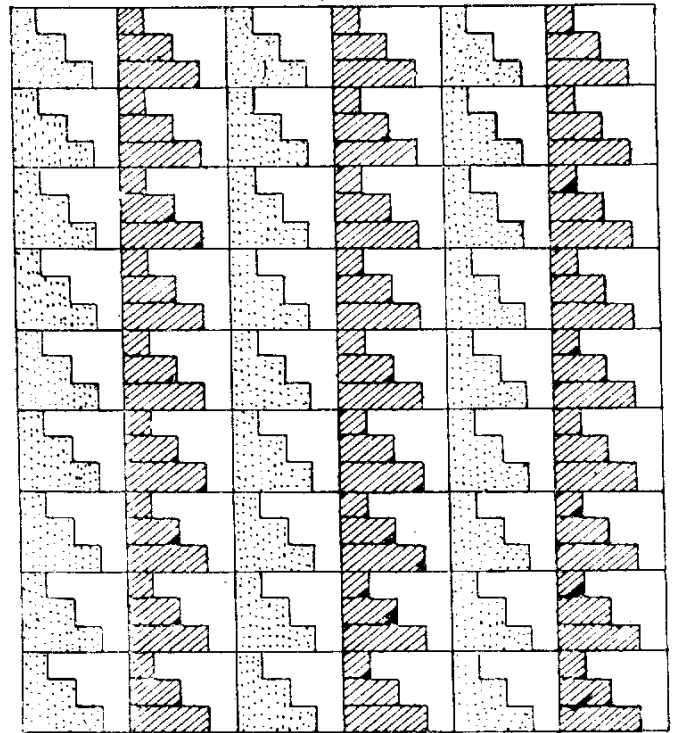
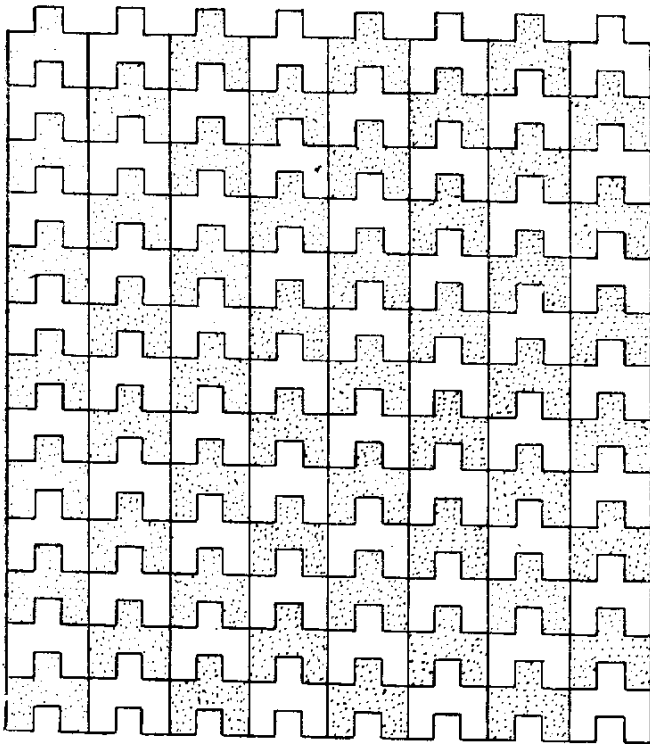
Let now the children check whether they can 'tessellate' a floor with any triangle, any quadrilateral, any pentagon, any hexagon, any heptagon, any octagon and any circle. They will succeed with the triangle, quadrilateral and hexagon only; with others, gaps will be left, not only at the edges but also in the interior.

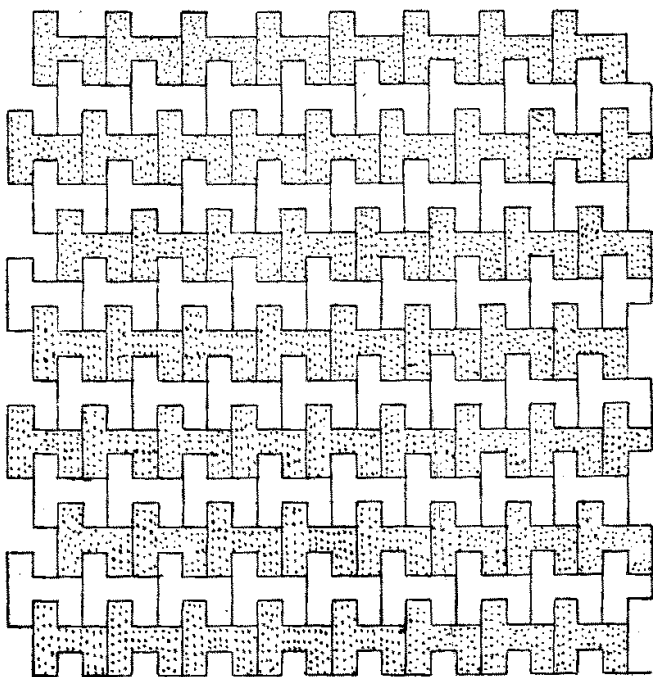




Experiment 201. Tessellations with Polyminoes.

Children have already made beautiful tessellation designs with polyminoes. Some other examples are given below :—





Experiment 202. Demonstrations of Theorems in Geometry.

Some of the theorems of plane geometry can be easily seen from the tessellations for triangles and quadrilaterals. Examples are :

- (a) the sum of angles of a triangle is two right angles.
- (b) the sum of the angles of a quadrilateral is four right angles.
- (c) if two straight lines intersect, vertically opposite angles are equal.
- (d) if two parallel lines are intersected by a third line, the alternate angles so formed are equal, the corresponding angles are equal, and the sum of the internal angles on one side is two right angles.
- (e) the line joining the middle points of two sides of a triangle is half of the base and is parallel to the base.

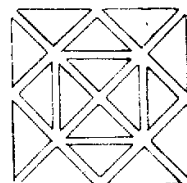
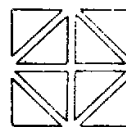
- (f) if angles of two triangles are equal each to each, then the corresponding sides are proportional.
- (g) if a number of parallel lines are intercepted by two transversals and if the intercepts on one of the transversals are equal, then the intercepts on the other transversal are also equal.

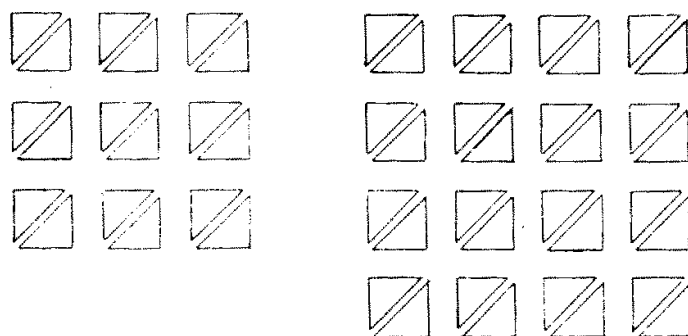
Experiment 203. Suitable Unit of Area.

To measure the area of a floor, we can use a triangle or a quadrilateral (square, rectangle, parallelogram, trapezium) or a hexagon as a unit and express the area as so many triangles or quadrilaterals or hexagons of given size. It is not possible to express it as so many pentagons or octagons or circles. The children may be given some triangle or quadrilateral or hexagon as a standard unit and asked to measure the area of the surface of the desk in term of these. The number of units is approximate since some area near the edges has to be left out. In practice we use a square (a square inch or square centimetre or a square foot or a square yard) as a unit. A square has an advantage in as much as it gives exact areas for square and rectangular fields, but if we have to measure the area of an equilateral triangular field, an equilateral triangle may be more convenient unit.

Experiment 204. Doubling the Area.

Children may be given a number of isoscles right-angled triangles of side say 2 cms. and asked to form squares with their help. They get the following figures :





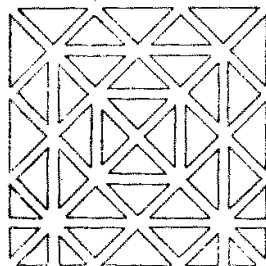
The number of triangles is 2, 4, 8, 16,
18, 32, 36,.....

Can the children find the pattern ?

Areas are 4, 8, 16, 32, 64, 72,.....

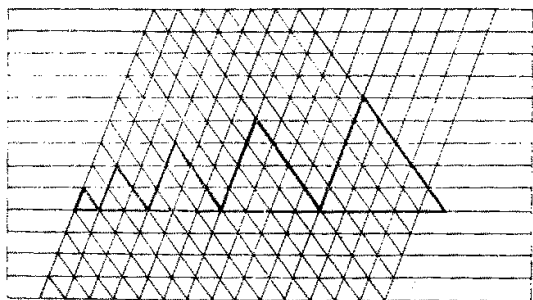
These are 2^2 , $(2\sqrt{2})^2$, 4^2 , $(4\sqrt{2})^2$, 6^2 ,
 $(6\sqrt{2})^2$, 8^2 , $(8\sqrt{2})^2$,.....

Given any square in the series, we
can find square of double its area.



Experiment 205. Similar Triangle Theorem.

The corresponding sides of the triangles are parallel and their
lengths are proportional.



Experiment 206. Even Numbers as Sums of Two Odd Primes.

Except the number 2, all prime numbers are odd and the sum
of every two odd prime numbers is even. What about the converse of
this result ? Can every even number be expressed as the sum of two
odd primes ? Let the children investigate. Let them first prepare
the table of all odd primes less than one hundred viz. 3, 5, 7, 11, 13,
17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83,
89, 97 by using sieve of Eratosthenes or otherwise. Let them proceed
to express every even number as the sum of two odd prime numbers
in as many ways as possible. They get

$$2=1+1, 4=1+3=2+2, 6=1+5=3+3.$$

However 1 is not a prime number and 2 is not an odd prime number.

Let the children proceed further :

$$8=3+5$$

$$10=3+7=5+5$$

$$12=5+7$$

$$14=3+11=7+7$$

$$16=3+13=5+11$$

$$18=5+13=7+11$$

$$\begin{aligned}
20 &= 3 + 17 = 7 + 13 \\
22 &= 3 + 19 = 5 + 17 = 11 + 11 \\
24 &= 5 + 19 = 7 + 17 = 11 + 13 \\
26 &= 3 + 23 = 7 + 19 = 13 + 13 \\
28 &= 5 + 23 = 11 + 17 \\
30 &= 7 + 23 = 11 + 19 = 13 + 17 \\
32 &= 3 + 29 = 13 + 19 \\
34 &= 3 + 31 = 11 + 23 = 17 + 17 \\
36 &= 5 + 31 = 7 + 29 = 13 + 23 = 17 + 19 \\
38 &= 7 + 31 = 19 + 19 \\
40 &= 3 + 37 = 11 + 29 = 17 + 23 \\
42 &= 5 + 37 = 13 + 29 = 19 + 23 = 11 + 31 \\
44 &= 3 + 41 = 7 + 37 = 13 + 31 \\
46 &= 3 + 43 = 5 + 41 = 17 + 29 = 23 + 23 \\
48 &= 5 + 43 = 7 + 41 = 17 + 31 = 19 + 29 = 11 + 37 \\
50 &= 3 + 47 = 7 + 43 = 13 + 37 = 19 + 31 \\
52 &= 5 + 47 = 11 + 41 = 23 + 29 \\
54 &= 7 + 47 = 11 + 43 = 13 + 41 = 17 + 37 = 23 + 31 \\
56 &= 3 + 53 = 13 + 43 = 19 + 37 \\
58 &= 5 + 53 = 11 + 47 = 17 + 41 = 29 + 29 \\
60 &= 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 29 + 31 \\
62 &= 3 + 59 = 19 + 43 = 31 + 31 \\
64 &= 3 + 61 = 5 + 59 = 11 + 53 = 17 + 47 = 23 + 41 \\
66 &= 5 + 61 = 7 + 59 = 13 + 53 = 23 + 43 = 29 + 37 = 19 + 47 \\
68 &= 7 + 61 = 31 + 37 \\
70 &= 3 + 67 = 11 + 59 = 17 + 53 = 23 + 47 = 29 + 41 \\
72 &= 5 + 67 = 11 + 61 = 13 + 59 = 19 + 53 = 29 + 43 = 31 + 41 \\
74 &= 3 + 71 = 7 + 67 = 13 + 61 = 31 + 43 = 37 + 37 \\
76 &= 3 + 73 = 5 + 71 = 17 + 59 = 23 + 53 = 29 + 47 \\
78 &= 5 + 73 = 7 + 71 = 11 + 67 = 17 + 61 = 19 + 59 = 31 + 47 = 37 + 41 \\
80 &= 7 + 73 = 13 + 67 = 19 + 61 = 37 + 43 \\
82 &= 3 + 79 = 11 + 71 = 23 + 59 = 29 + 53 = 41 + 41 \\
84 &= 5 + 79 = 11 + 73 = 13 + 71 = 17 + 67 = 23 + 61 \\
&\quad = 31 + 53 = 37 + 47 = 41 + 43 \\
86 &= 3 + 83 = 7 + 79 = 13 + 73 = 19 + 67 = 43 + 43 \\
88 &= 5 + 83 = 17 + 71 = 29 + 59 = 41 + 47 \\
90 &= 7 + 83 = 11 + 79 = 17 + 73 = 19 + 71 = 29 + 61 \\
&\quad = 37 + 53 = 43 + 47 = 23 + 67 = 31 + 59 \\
92 &= 3 + 89 = 13 + 79 = 19 + 73 = 31 + 61 \\
94 &= 5 + 89 = 11 + 83 = 23 + 71 = 41 + 53 = 47 + 47
\end{aligned}$$

$$96 = 7 + 89 = 13 + 83 = 17 + 79 = 29 + 67 = 37 + 59 = 43 + 53 = 23 + 73$$

$$98 = 19 + 79 = 31 + 67 = 37 + 61$$

$$100 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 = 3 + 97.$$

The children have been able to express all even numbers 14 or greater as the sum of two odd prime numbers in more than one way. Will this always be possible? It appears very likely but nobody has proved this result so far. Will the children accept the challenge?

Experiment 207. Numbers which can be expressed as product of a given number of numbers in only one way.

Every composite number can be expressed, as the product of prime factors. Let the children verify that they always get the same prime factors *i.e.*, in any two prime factorizations, the factors will be the same, except possibly for the order of the factors.

Let them find those numbers which are products of only two factors, prime or composite in only one way *e.g.*, 4, 6, 8, 9, 10, 14, 15, 21, 22, 25, 26, 27, 33, 34, 35, 38, 39,..... What pattern do they find? The numbers are either the product of two prime factors or they are perfect cubes of prime numbers. All other composite numbers can be expressed as the product of two factors in more than one way.

Let the children now find all those numbers which can be expressed as the product of exactly three numbers in one way only. These will be products of three prime numbers or perfect fourth powers of prime numbers.

The children can now try to generalise and find all those numbers which can be expressed as the products of exactly n numbers in one way only. These will be either products of n prime numbers or these will be perfect $(n+1)$ th power of prime numbers.

Experiment 208. Maximum Product with a Given Sum.

Let the children be given a number, say 16 or 17 and let them break it into 2 addends so that the product is maximum $1 \times 15 = 15$, $2 \times 14 = 28$, $3 \times 13 = 39$, $4 \times 12 = 48$, $5 \times 11 = 55$, $6 \times 10 = 60$, $7 \times 9 = 63$, $8 \times 8 = 64$, $1 \times 16 = 16$, $2 \times 15 = 30$, $3 \times 14 = 42$, $4 \times 13 = 52$, $5 \times 12 = 60$, $6 \times 11 = 66$, $7 \times 10 = 70$, $8 \times 9 = 72$, so that the product is maximum when the two numbers are either equal or as near to each other as possible.

Next children break the given number 16 into 3 addends so that the product is maximum. The maximum for 16 is $5 \times 5 \times 6 = 150$ and this maximum is greater than the earlier maximum.

Again let them break 16 into 4 or 5 or 6 or 7 addends and see what they get

$$\begin{array}{ll} 4+4+4+4=16 & 4 \times 4 \times 4 \times 4=256 \\ 3+3+3+3+2+2=16 & 3 \times 3 \times 3 \times 3 \times 2 \times 2=324 \\ 3+3+3+3+4=16 & 3 \times 3 \times 3 \times 3 \times 4=324 \\ 3+3+3+3+3+1=16 & 3 \times 3 \times 3 \times 3 \times 3 \times 1=243 \end{array}$$

It appears that the largest product which we can get from numbers whose sum is 16 is 324.

Let the children investigate other numbers and get the following results :

$$\begin{array}{llll} 2+3+3+3 & =11 & 2 \times 3 \times 3 \times 3 & =54 \\ 3+3+3+3 & =12 & 3 \times 3 \times 3 \times 3 & =81 \\ 4+3+3+3 & =13 & 4 \times 3 \times 3 \times 3 & =108 \\ 2+3+3+3+3 & =14 & 2 \times 3 \times 3 \times 3 \times 3 & =162 \\ 3+3+3+3+3 & =15 & 3 \times 3 \times 3 \times 3 \times 3 & =243 \\ 4+3+3+3+3 & =16 & 4 \times 3 \times 3 \times 3 \times 3 & =324 \\ 2+3+3+3+3+3 & =17 & 2 \times 3 \times 3 \times 3 \times 3 \times 3 & =486 \end{array}$$

It appears that if the given number is $3n$, then the maximum product is 3^n , if the given number is $3n+1$, then the maximum product is $3^{n-1} \times 4$ and if the given number is $3n+2$, then the maximum product is $3^n \times 2$. Let the children verify whether this pattern holds for numbers upto 50.

The problem of minimum product with a given sum is of no interest since the minimum product is always unity.

Experiment 209. Maximum Product with Given Sum and Different Addends.

In the last section, we found 3 as the most desirable addend. We took as many three's as possible and then took 2 or 4 wherever necessary. Since here we want different addends, it is desirable to consider 2, 3, 4, 5, 6 as addends. Let the children now verify the following :

$$\begin{array}{llll} 2+3 & =5 & 3 \times 2 & =6 \\ 2+4 & =6 & 2 \times 4 & =8 \\ 3+4 & =7 & 3 \times 4 & =12 \\ 3+5 & =8 & 3 \times 5 & =15 \\ 2+3+4 & =9 & 2 \times 3 \times 4 & =24 \end{array}$$

$$\begin{array}{llll} 2+3+5 & =10 & 2 \times 3 \times 5 & =30 \\ 2+4+5 & =11 & 2 \times 4 \times 5 & =40 \\ 3+4+5 & =12 & 3 \times 4 \times 5 & =60 \\ 3+4+6 & =13 & 3 \times 4 \times 6 & =72 \\ 2+3+4+5 & =14 & 2 \times 3 \times 4 \times 5 & =120 \\ 2+3+4+6 & =15 & 2 \times 3 \times 4 \times 6 & =144 \\ 2+3+5+6 & =16 & 2 \times 3 \times 5 \times 6 & =180 \\ 2+4+5+6 & =17 & 2 \times 4 \times 5 \times 6 & =240 \\ 3+4+5+6 & =18 & 3 \times 4 \times 5 \times 6 & =360 \end{array}$$

Let the children continue the pattern for numbers upto 50.

Experiment 210. Other similar problems.

Children may be asked to investigate the following problems and note the emerging patterns :

- Break up numbers from 11 to 50 into prime addends so that the product is maximum.
- Break up numbers from 11 to 50 into factors so that the sum of the factors is minimum.
- Break up numbers from 11 to 50 into addends so that the sum of squares of numbers is maximum (or minimum).

The children may be asked to construct more such problems and to investigate them. These problems will provide the children with well motivated exercises in addition and multiplication.

Experiments: 211 to 225

Experiment 211. Numbers have many names.

Let the children express a given number in as many different ways as possible by using fundamental operations, e.g.,

$$\begin{aligned}
 16 &= 1 + 15 = 7 + 9 = 5 + 11 = 3 + 13 = \dots\dots\dots \\
 &= 17 - 1 = 18 - 2 = 19 - 3 = 25 - 9 = \dots\dots\dots \\
 &= 1 \times 16 = 2 \times 8 = 4 \times 4 = \dots\dots\dots \\
 &= 32 \div 2 = 64 \div 4 = 192 \div 12 = \dots\dots\dots \\
 &= (4 + 4) + (4 \times 2) = (32 \div 8) + (4 \times 5) - 8 = \dots\dots\dots \\
 &= \dots\dots\dots
 \end{aligned}$$

Let each child express each number by using one operation, two operations, three operations and all the four operations.

Experiment 212. Expressing every number in terms of four three's or four four's or four five's.

In another version children may try to express every number in terms of four three's only or four four's only or four five's only, e.g., we have

$$\begin{array}{llll}
 (a) \quad 3 + 3 - 3 - 3 & = 0 & (3 \div 3) \times (3 \div 3) & = 1 \\
 (3 \div 3) + (3 \div 3) & = 2 & (3 + 3 + 3) \div 3 & = 3
 \end{array}$$

$$\begin{array}{llll}
 \frac{3}{3} - (3 + 3) & = 4 & (3 + 3) - (3 \div 3) & = 5 \\
 3 + 3 + 3 - 3 & = 6 & 3 + 3 + \frac{3}{3} & = 7 \\
 (33 \div 3) - 3 & = 8 & (3 \times 3) + 3 - 3 & = 9 \\
 (33 - 3) \div 3 & = 10 & \frac{3}{3} + \frac{3}{3} & = 11 \\
 (33 + 3) \div 3 & = 12 & \frac{3}{3} + \sqrt{3 \times 3} & = 13 \\
 33 \div 3 + 3 & = 14 & (3 \times 3) + (3 + 3) & = 15
 \end{array}$$

$$\begin{array}{llll}
 (b) \quad (4 + 4) - (4 + 4) & = 0 & \frac{4}{4} & = 1 \\
 \frac{4}{4} + \frac{4}{4} & = 2 & \frac{4}{4} + 4 - \sqrt{4} & = 3 \\
 4 \div 4 - \sqrt{4} - \sqrt{4} & = 4 & \frac{4}{4} + \sqrt{4} + \sqrt{4} & = 5 \\
 \sqrt{4} + \sqrt{4} + 4 - \sqrt{4} & = 6 & \frac{4}{4} + 4 + \sqrt{4} & = 7 \\
 \sqrt{4} + \sqrt{4} + \sqrt{4} + \sqrt{4} & = 8 & 4 + 4 + \frac{4}{4} & = 9 \\
 4 + \sqrt{4} + \sqrt{4} + \sqrt{4} & = 10 & \frac{44}{\sqrt{4} + \sqrt{4}} & = 11 \\
 4 + 4 + \sqrt{4} + \sqrt{4} & = 12 & \frac{4}{4} + \sqrt{4} & = 13 \\
 4 + 4 + 4 + \sqrt{4} & = 14 & \frac{4}{4} + 4 & = 15 \\
 (c) \quad (5 + 5) - (5 + 5) & = 0 & \frac{5}{5} & = 1 \\
 \frac{5}{5} + \frac{5}{5} & = 2 & \frac{5 + 5 + 5}{5} & = 3 \\
 \sqrt{5 \times 5} - 5 \div 5 & = 4 & 5 \times 5 + 5 \times 5 & = 5 \\
 \frac{5}{5} - 5 & = 6 & 5 + (5 \times 5) - 5 & = 7 \\
 5 + (5 \times 5) + 5 & = 8 & \frac{5}{5} - \frac{5}{5} & = 9 \\
 5 + 5 + 5 - 5 & = 10 & \frac{5}{5} + \frac{5}{5} & = 11 \\
 (d) \quad \frac{a}{a} \times \frac{a}{a} & = 1 & \frac{a}{a} + \frac{a}{a} & = 2 \\
 \frac{a + a + a}{a} & = 3 & \frac{a}{a} - \frac{a}{a} & = 9 \\
 \frac{a}{a} \times \frac{a}{a} & = 10 & \frac{a}{a} + \frac{a}{a} & = 11 \\
 \frac{a}{a} + \frac{a}{a} & = 20 & &
 \end{array}$$

Experiment 213. Laws of Algebraic Structure.

$b+a$ is another name for $a+b$	(commutative law for addition)
$b \times a$ is another name for $a \times b$	(commutative law for multiplication)
$a+(b+c)$ is another name for $(a+b)+c$	(associative law for addition)
$a \times (b \times c)$ is another name for $(a \times b) \times c$	(associative law for multiplication)
$a \times b + a \times c$ is another name for $a \times (b+c)$	(multiplication distributes addition on the right)
$b \times a + c \times a$ is another name for $(b+c) \times a$	(multiplication distributes addition on the left)
$(b+c) \div a$ is another name for $b \div a + c \div a$	(division distributes addition on the left)
$a-b$ is not another name for $b-a$	(commutative law does not hold for subtraction)
$a \div b$ is not another name for $b \div a$	(commutative law does not hold for division)
$a \div (b+c)$ is not another name for $a \div b + a \div c$	(division does not distribute addition on the right)
$a \times (b-c)$ is another name for $a \times b - a \times c$	(multiplication distributes subtraction)

Above and similar laws can be profusely illustrated by examples of this game.

Experiment 214. What is my Rule?

A child calls out a number and teacher calls out a corresponding number. A second child then calls out another number and the teacher calls out another corresponding number and this continues for four or five children *e.g.* we may get

Children's numbers	3	8	15	19	...	...	...
Teacher's numbers	5	10	17	21	...	...	...

The teacher now asks "What is my rule?" and the children have no difficulty in saying that his rule is addition of 2 to the child's number. If a child calls out x , he calls out $x+2$.

The teacher can now follow a large number of rules *e.g.*

$$\begin{aligned}x &\rightarrow x+5 \\x &\rightarrow x-1 \\x &\rightarrow 2x+1\end{aligned}$$

$$\begin{aligned}x &\rightarrow 3x-2 \\x &\rightarrow x^2 \\x &\rightarrow x^2+1 \\x &\rightarrow x^2-1 \\x &\rightarrow x^3 \\x &\rightarrow x^2+x\end{aligned}$$

and in each case the children have to guess his rule.

After this, each child makes a rule and other children guess it. If the rule is $x \rightarrow x-1$ and some child says 0, the teacher can give the answer '-1' if the children know negative integers, otherwise he simply asks the child to give another bigger number.

Experiment 215. What is my Machine?

Next a child calls out pairs of numbers and the teacher gives out corresponding number and the children guess the rule *e.g.*

Children's pair of numbers	3,4	3,5	4,6	5,7	8,9
Teacher's numbers	7	8	10	12	17

Children easily see that the teacher's rule is "Add the numbers given by the child".

In this case if a child calls out (x, y) , the teacher calls out $x+y$. In the same manner the teacher can follow the rules:

Child's pair of numbers	Teacher's number
$(x, y) \rightarrow$	$x+y$
$(x, y) \rightarrow$	$x \times y$
$(x, y) \rightarrow$	$ x-y $ (<i>i.e.</i> difference of x and y)
$(x, y) \rightarrow$	$x-y$ (if the children know negative integers)
$(x, y) \rightarrow$	$x \div y$ (if the children know fractions)
$(x, y) \rightarrow$	$x+2y$
$(x, y) \rightarrow$	$y+2x$
$(x, y) \rightarrow$	x^2+y^2
$(x, y) \rightarrow$	$(x-y)^2$

In each case the children guess the rule. Later each child makes his own rule and others guess the rule.

This game is called "What is my machine?" If the teacher has chosen $x+y$, the child says he is using the addition machine. The other machines which the teacher can use are

“multiplication machine”, “subtraction machine”, “difference machine”, “division machine”, “square of difference machine”, “squaring and summing machine”, etc.

Experiment 216. What are the two Machines?

Next a child calls out a triplet of numbers and the teacher gives the corresponding number. The rules which the teacher can follow are.

(x, y, z)	$\rightarrow$	$x+y+z$
(x, y, z)	$\rightarrow$	$x \times y \times z$
(x, y, z)	$\rightarrow$	$x+y-z$
(x, y, z)	$\rightarrow$	$x-y+z$
(x, y, z)	$\rightarrow$	$(x \times y) + z$
(x, y, z)	$\rightarrow$	$x + (y \times z)$

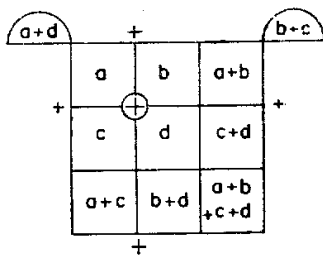
Here the teacher is using, two machines in succession like ‘addition-addition’, ‘multiplication-multiplication’, ‘addition-subtraction’ etc. It will be seen that in most cases, the order in which machines are used is important.

Since two machines are used, it may not be easy for the children to guess the rule. It may be useful for the children to have first practice in direct use of rules.

The object of the last three games is to give the children some insight into the concepts of correspondence, mapping and functions of one, two and three variables.

Experiment 217. Cross Number Game: Addition along Rows, Columns and Diagonals.

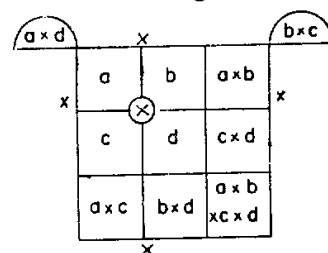
We are given four numbers, a, b, c, d arranged in two rows and two columns. The two rows are added separately. Children may be surprised to find at first that the sum of the row-sums is always equal to the sum of the column-sums, but later they should be able to understand why it is so. Children can do a large number of numerical examples of this type. In the semi-circles, we have written



sums of diagonal elements. The sum of these two diagonal sums also comes out to be same as the sum of two row-sums or column-sums.

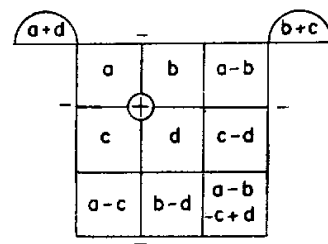
Experiment 218. Cross Numbers Game: Multiplication along Rows, Columns and Diagonals.

Here elements of each row are multiplied and then elements of each column are multiplied. The products of two row-products and of two column-products and of two diagonal-products will again come out to be equal.



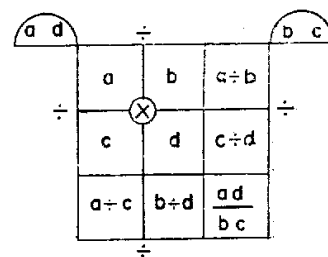
Experiment 219. Cross Number Game: Subtraction along Rows and Columns, Addition along Diagonals.

Here we subtract second element of a row from the first and then subtract the second row-sum from the first row-sum. We do the same for columns. For diagonals we first add diagonal elements and then subtract the diagonal sums. We find the difference of row-differences = difference of column-differences = difference of diagonal sums. If the children do not know negative integers, the numbers a, b, c, d should be carefully chosen.



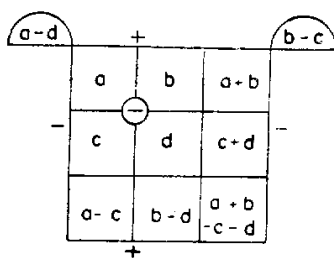
Experiment 220. Cross Number Game: Division along Rows and Columns and Multiplication along Diagonals.

Here we replace $-$ and $+$ of last experiment by $\div$ and $\times$. If the children do not know fractions, the numbers have to be chosen carefully.



Experiment 221. Cross Number Game : Addition along Rows and Subtraction along Columns and Diagonals

Here we use addition along rows, subtraction along columns and subtraction along diagonals.



Experiment 222. Other Possible Cross Number Games.

We have four operations viz.

$$+, -, \times, \div$$

and we can use these along rows, along columns and along diagonals.

Along rows we can choose the operation in 4 ways.

Along columns, we can choose the operation in 4 ways.

Along diagonals, we can choose the operation in 4 ways.

There are thus 64 possibilities, all of which are given below as 64 triplets.

The three members of the triplet give row, column and diagonal operations respectively. Let the children be encouraged to write all the 64 triplets themselves in a systematic way and then to find out which are feasible and which are not. The children can write first those 16 triplets in which the row operation is addition, then those 16 triplets in which the row operation is subtraction, then those 16 triplets in which the row operation is multiplication and then finally they can write those 16 triplets in which the row operation is division.

Each set of 16 triplets will be divided into four subsets of four each in which the column operation will be addition, subtraction, multiplication and division.

$(-++)$	$(++-)$	$(++\times)$	$(++\div)$
$(+-+)$	$(+--)$	$(+-\times)$	$(+-\div)$
$(\div\times+)$	$(+\times-)$	$(+\times\times)$	$(+\times\div)$
$(+\div+)$	$(+\div-)$	$(+\div\times)$	$(+\div\div)$
$(-++)$	$(-+-)$	$(--\times)$	$(--\div)$
$(-+-)$	$(---)$	$(--\times)$	$(--\div)$
$(-\times+)$	$(-\times-)$	$(-\times\times)$	$(-\times\div)$
$(-\div+)$	$(-\div-)$	$(-\div\times)$	$(-\div\div)$

$(\times++)$	$(\times+-)$	$(\times+\times)$	$(\times+\div)$
$(\times-+)$	$(\times--)$	$(\times-\times)$	$(\times-\div)$
$(\times\times+)$	$(\times\times-)$	$(\times\times\times)$	$(\times\times\div)$
$(\times\div+)$	$(\times\div-)$	$(\times\div\times)$	$(\times\div\div)$
$(\div++)$	$(\div+-)$	$(\div+\times)$	$(\div+\div)$
$(\div-+)$	$(\div--)$	$(\div-\times)$	$(\div-\div)$
$(\div\times+)$	$(\div\times-)$	$(\div\times\times)$	$(\div\times\div)$
$(\div\div+)$	$(\div\div-)$	$(\div\div\times)$	$(\div\div\div)$

Children have to find out as to which triplets are permissible.

Children may choose numbers and try out. If the pattern fails in one case, they can write 'No'. If however it succeeds, it may be better to try for at least one more example before writing 'Yes'.

Those children who have learnt some algebra can complete the table more easily and it will give them sufficient practice in algebraic manipulations.

Experiment 223. Game of Clues

(i) Suppose the teacher says he is thinking of a partition of some number. This gives the child some clue of the type of problem, but still there are an infinite number of possibilities open. The teacher gives one more clue viz. that he is thinking of partition of number 4. This gives substantial information in that it reduces the number of possibilities to just eight i.e. the teacher could have been thinking of 4 or 3+1 or 2+2 or 1+3 or 2+1+1 or 1+2+1 or 1+1+2 or 1+1+1+1. If the teacher gives one more clue viz. that he is thinking of a partition into 3 parts, it gives some additional information since it reduces the number of possibilities to 3 viz. 2+1+1, 1+2+1, 1+1+2. If he now gives another clue viz. he is thinking of a partition starting with 1, it reduces the number of possibilities to two viz. 1+2+1 and 1+1+2 and if he finally says he is thinking of the partition ending with 2, there is only one possibility left viz. 1+1+2. At each stage we are getting additional information. This information can be measured and the subject which discusses this measurement of information is called theory of information, but we shall not discuss it here. However if the teacher had said that he was thinking of a partition of an even number, it would have given less information than was given by the second clue above. If his third clue had been that the number of components was not more than four, it would have given no information, as the

number of possibilities would have still remained eight. The teacher can encourage the children to ask for clues *e.g.* he can say "I am thinking of a partition of number 5. Now you should ask the smallest number of questions in order to know the partition."

(ii) Try similar questions *e.g.* "Think of a number.", "Is it even or odd?" "Is it prime or not?", "Is it less than hundred or not?"

(iii) There are twelve coins, one of which is defective and heavier than other. We have to get this information in as few weighings as possible with an equal arms balance. We put four coins in each pan. One weighing will reduce the possibilities from 12 to 4. We now put one coin in each pan and weigh. This will reduce the possibilities to one or two. If it reduces to two, one more weighing will be necessary. Let the children now argue similarly the cases for 15 coins, 21 coins and so on.

(iv) Similarly if the teacher says "I am thinking of a word, now you start asking questions". The children can ask "How many letters has it?" "What is the first letter, what is the second letter and so on?" At each stage the question asked should be such as to reduce the number of possibilities as much as possible.

Experiment 224. Story-Telling Games

Story problems or 'word problems' are usually a source of trouble to the children. They constitute a test of a child's reading and understanding ability as much as of his mathematical ability. This activity is meant to relate the abstractions of mathematics to the problems of the real world.

Instead of giving the children word problems, it may be useful to ask them to simultaneously invent stories to fit arithmetic problems *e.g.*, for

$$25 - 18 = 7$$

the children may give the stories.

- (i) I had to go 25 yards, I have gone 18 yards; I have still to go 7 yards.
- (ii) I have two brothers 25 and 18 years of age. My elder brother is 7 years older than the younger one.
- (iii) I deposited Rs. 25 in the bank, I have withdrawn Rs. 18, then Rs. 7 remain in my account.

Similarly for

$$4 \times 5 = 20$$

the children may give the stories:

- (i) Five groups of four ducks each give twenty ducks.
- (ii) I got four marks in each of five questions, I got 20 marks.
- (iii) The father gave five rupees to each of his four sons. He gave twenty rupees.

For

$$4 \times 5 + 3 = 23$$

the children may give stories of the type:

- (i) I bought 4 balloons of five paise each and one toy elephant of three paise; I spent twenty-three paise in all.
- (ii) The teacher divided the children into five groups of 4 each and three boys were left out. The total number of children was twenty-three.

For

$$20 \div 4 = 5$$

the children may give a story of the type:

The teacher divided twenty children into groups of four each. Five groups were formed.

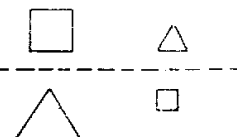
This activity of telling stories can be quite popular with the children. The teacher can give hints where necessary.

Experiment 225. The game of fewest questions.

This is quite popular with the children, strengthens their number concepts, expands their vocabulary and develops powers of reasoning.

We illustrate it below:

- (i) I am thinking of one of these for shapes. Ask questions to find which one is it. A child may ask three questions *viz.* Is it a large square? Is it a small triangle? Is it a large triangle? But it is obvious



- that he can get the answer in two questions and the children can be asked to find many such pairs, *e.g.*,
 (a) Is it above the line? Is it a square? (b) Is it above the line? Is it large? (c) Is it small? Is it a triangle? etc.
 (ii) I am thinking of the numbers upto 16. Now ask minimum number of questions to find which number I am

13 Patterns with Cuisenaire Rods

Experiments : 226 to 236

thinking? One child may ask : Is it 1? Is it 2? Is it 3? If he is unlucky, he may have to ask fifteen questions. If he is lucky, one question may do. However we want the minimum number of questions to ensure that the child knows the answer *e.g.* the question may be :

Is it even or odd? Is it ≤ 8 ? Is it ≤ 4 ? (*or*, Is it ≤ 12)

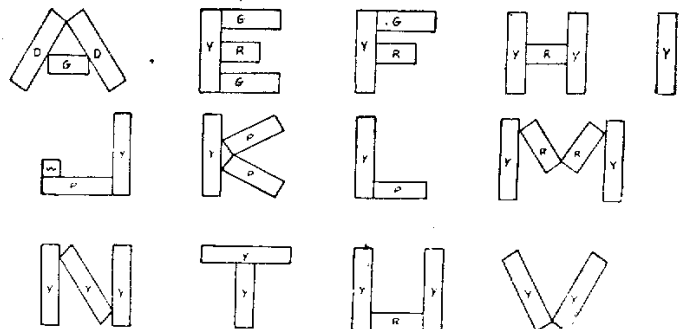
One more question will be necessary.

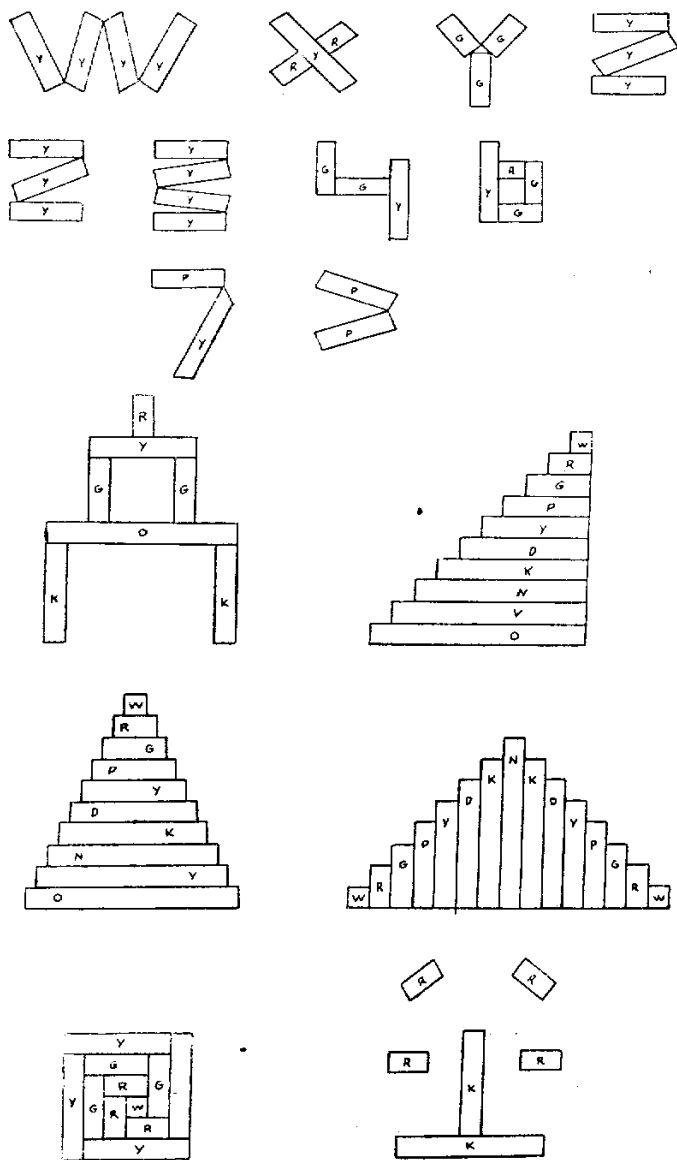
At every stage the number of possibilities is reduced by half.

- (iii) The children may find each other's year of birth or month of birth or date of birth by asking minimum number of questions. This game is quite similar to game of clues and is connected with information theory.

Experiment 226. Free-Play Activities.

We have ten rods *viz.*, white (W), red (R), green (G), purple (P), yellow (Y), dark green (D), black (K), brown (N), blue (U), and orange (O) of lengths 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 units respectively. First children may build letters of the alphabet, houses, trains, floors, symbols, gardens, towers, etc. *e.g.*





Experiment 227. Some Properties of Rods.

The children may be helped to discover the following facts. All rods are rectangular, all rods of same colour are equal in length, ten different colours are used for rods, white rod is shortest, orange rod is longest, green rod is shorter than dark green rod, each rod has 6 faces, 8 vertices and 12 edges, white rods are cubes, other rods are cuboids, vertical structures constructed with small rods are more stable than those constructed with larger rods, trains of rods can be constructed by placing the rods end to end, the positions of rods in a train can be changed without changing the length of the train, some trains are equal in length to other trains, all trains are not of equal length, all rods are multiples of white rods, some rods (red, purple, dark green, brown and orange) are multiples of red rod, some rods (light green, dark green and blue) are multiples of light green rod, some rods (purple and brown) are multiples of purple rod, some rods (yellow and orange) are multiples of yellow rod, floor of rods can be formed by placing rods side by side, rods of same colour can be used to construct cubes, rods of different colours can be used to make staircases.

Experiment 228. Addition and Subtraction Facts.

There are 45 addition and 45 subtraction facts for numbers 1 to 10. All these can be illustrated with the rods:

Rod Arrangement	Addition Fact	Subtraction Fact
	$1 + 1 = 2$	$2 - 1 = 1$
	$2 + 1 = 3$	$3 - 1 = 2$
	$1 + 2 = 3$	$3 - 2 = 1$
	$3 + 1 = 4$	$4 - 1 = 3$
	$1 + 3 = 4$	$4 - 3 = 1$
	$2 + 2 = 4$	$4 - 2 = 2$
	$4 + 1 = 5$	$5 - 1 = 4$
	$3 + 2 = 5$	$5 - 2 = 3$
	$2 + 3 = 5$	$5 - 3 = 2$
	$1 + 4 = 5$	$5 - 4 = 1$

Symmetry in some of the patterns may be noted.

D	
Y	W
P	R
G	G
R	P
W	Y

$$\begin{aligned} 5 + 1 &= 6 \\ 4 + 2 &= 6 \\ 3 + 3 &= 6 \\ 2 + 4 &= 6 \\ 1 + 5 &= 6 \end{aligned}$$

$$\begin{aligned} 6 - 1 &= 5 \\ 6 - 2 &= 4 \\ 6 - 3 &= 3 \\ 6 - 4 &= 2 \\ 6 - 5 &= 1 \end{aligned}$$

With the help of the rods, the children can illustrate the commutative law for addition.

G	R
R	G

Y	R
R	Y

K	
D	W
Y	R
P	G
G	P
R	Y
W	D

$$\begin{aligned} 6 + 1 &= 7 \\ 5 + 2 &= 7 \\ 4 + 3 &= 7 \\ 3 + 4 &= 7 \\ 2 + 5 &= 7 \\ 1 + 6 &= 7 \end{aligned}$$

$$\begin{aligned} 7 - 1 &= 6 \\ 7 - 2 &= 5 \\ 7 - 3 &= 4 \\ 7 - 4 &= 3 \\ 7 - 5 &= 2 \\ 7 - 6 &= 1 \end{aligned}$$

They can also illustrate the associative law for addition.

V		
R	G	P
R	G	P

N	
K	W
D	R
V	G
P	P
G	Y
R	D
W	K

$$\begin{aligned} 7 + 1 &= 8 \\ 6 + 2 &= 8 \\ 5 + 3 &= 8 \\ 4 + 4 &= 8 \\ 3 + 5 &= 8 \\ 2 + 6 &= 8 \\ 1 + 7 &= 8 \end{aligned}$$

$$\begin{aligned} 8 - 1 &= 7 \\ 8 - 2 &= 6 \\ 8 - 3 &= 5 \\ 8 - 4 &= 4 \\ 8 - 5 &= 3 \\ 8 - 6 &= 2 \\ 8 - 7 &= 1 \end{aligned}$$

Experiment 229. Partitions with Cuisenaire Rods.

As shown in experiment 228, all partitions of number upto ten into non-zero addends can be expressed through Cuisenaire rods e.g.

V	
N	W
K	R
D	G
Y	P
P	Y
G	D
R	K
W	N

$$\begin{aligned} 8 + 1 &= 9 \\ 7 + 2 &= 9 \\ 6 + 3 &= 9 \\ 5 + 4 &= 9 \\ 4 + 5 &= 9 \\ 3 + 6 &= 9 \\ 2 + 7 &= 9 \\ 1 + 8 &= 9 \end{aligned}$$

$$\begin{aligned} 9 - 1 &= 8 \\ 9 - 2 &= 7 \\ 9 - 3 &= 6 \\ 9 - 4 &= 5 \\ 9 - 5 &= 4 \\ 9 - 6 &= 3 \\ 9 - 7 &= 2 \\ 9 - 8 &= 1 \end{aligned}$$

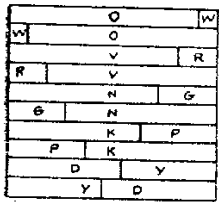
O	
V	W
N	R
K	G
D	P
Y	Y
P	D
G	K
R	N
W	V

$$\begin{aligned} 9 + 1 &= 10 \\ 8 + 2 &= 10 \\ 7 + 3 &= 10 \\ 6 + 4 &= 10 \\ 5 + 5 &= 10 \\ 4 + 6 &= 10 \\ 3 + 7 &= 10 \\ 2 + 8 &= 10 \\ 1 + 9 &= 10 \end{aligned}$$

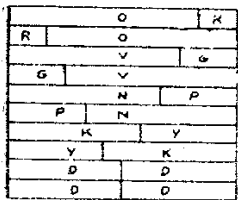
$$\begin{aligned} 10 - 1 &= 9 \\ 10 - 2 &= 8 \\ 10 - 3 &= 7 \\ 10 - 4 &= 6 \\ 10 - 5 &= 5 \\ 10 - 6 &= 4 \\ 10 - 7 &= 3 \\ 10 - 8 &= 2 \\ 10 - 9 &= 1 \end{aligned}$$

Similarly we may express the partitions of 11 to 20 in terms of 2 addends only as follows :

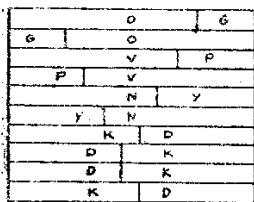
Y	
P	W
G	R
R	G
W	P
G	W
W	G
W	W
R	R
R	W
W	R
R	W
W	W
W	R
W	W
W	W



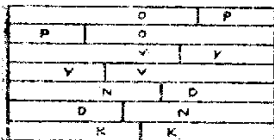
$$\begin{array}{llll}
 10+1 & 1+10 & 9+2 & 2+9 \\
 8+3 & 3+8 & 7+4 & 4+7 \\
 6+5 & 5+6 & &
 \end{array}$$



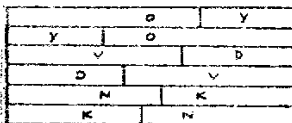
$$\begin{array}{llll}
 10+2 & 2+10 & 9+3 & 3+9 \\
 8+4 & 4+8 & 7+5 & 5+7 \\
 6+6 & 6+6 & &
 \end{array}$$



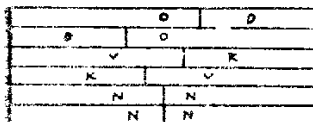
$$\begin{array}{llll}
 10+3 & 3+10 & 9+4 & 4+9 \\
 8+5 & 5+8 & 7+6 & 6+7 \\
 6+7 & 7+6 & &
 \end{array}$$



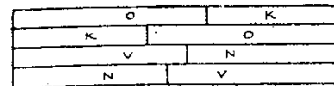
$$\begin{array}{llll}
 10+4 & 4+10 & 9+5 & 5+9 \\
 8+6 & 6+8 & 7+7 &
 \end{array}$$



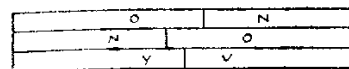
$$\begin{array}{llll}
 10+5 & 5+10 & 9+6 & 6+9 \\
 8+7 & 7+8 & &
 \end{array}$$



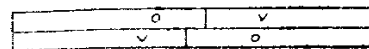
$$\begin{array}{llll}
 10+6 & 6+10 & 9+7 & 7+9 \\
 8+8 & & &
 \end{array}$$



$$\begin{array}{ll}
 10+7 & 7+10 \\
 9+8 & 8+9
 \end{array}$$



$$\begin{array}{ll}
 10+8 & 8+10 \\
 9+9 &
 \end{array}$$



$$\begin{array}{ll}
 10+9 & 9+10
 \end{array}$$



$$10+10$$

Experiment 230. Tens and Units.

There are three boxes. The first box contains rods of all lengths. The second box contains only orange rods (called tens) and the third box contains only white rods (called units).

The first child makes a train of length upto 50. The second child puts as many orange rods below this train as possible. The third child then completes the train with white rods.



The fourth child records the result as follows :

Train	Number of Orange Rods (Tens)	Number of White Rods (Units)	Sum
5+6+4+3+2+1	2	1	21
.....			

Children can form teams of four children each and each team can carry out a dozen or two of such measurements.

Experiment 231. Family of Differences Game with Cuisenaire Rods.

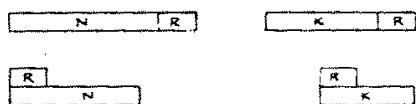
Children have to place one rod over another so that the difference is a specified number. Thus to get difference 4, the child may place a white rod over a yellow rod, a red rod over a dark green rod, a green rod over a black rod, a purple rod over a brown rod, a yellow rod over a blue rod and dark green rod over an orange rod, so that

$$4=5-1=6-2=7-3=8-4=9-5=10-6.$$

Similarly let the children find family of differences for 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Experiment 232. Sum and Difference Game with Cuisenaire Rods.

Let a child choose two rods, make a train out of these and also place these one above the other



and write the results as follows :

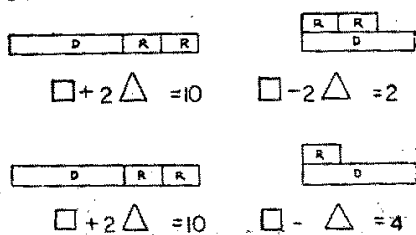
$$8+2=10, 8-2=6; \quad 7+2=9, 7-2=5.$$

Let different children do different problems and speak out their results which may be written on the board. After children have got a lot of practice in these, give them the converse problems, e.g.

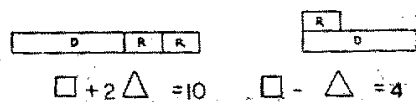
$$\begin{array}{ll} \square + \triangle = 9 & \square - \triangle = 1 \\ \square + \triangle = 15 & \square - \triangle = 3 \\ \square + \triangle = 15 & \square - \triangle = 6 \\ \square + \triangle = 12 & \square - \triangle = 3 \end{array}$$

In the last two cases, they do not get a solution. Why? In the earlier pattern, they had noticed that sum and difference of two numbers are either both even or odd. They may also notice that the number in $\square$ is half the sum and that in $\triangle$ is half the difference of two given numbers.

The children may consider the game with one large and two small rods, e.g.,



$$\square + 2\triangle = 10 \quad \square - 2\triangle = 2$$



$$\square + 2\triangle = 10 \quad \square - \triangle = 4$$

Experiment 233. Facts Team Game with Cuisenaire Rods.

An orange rod may be covered by a red and a brown rod and the children may read out the team of four facts viz.

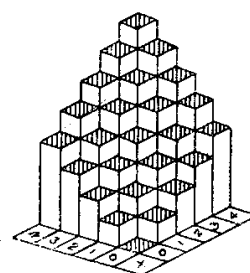
$$8+2=10, 2+8=10, 10-8=2, 10-2=8$$

Children may form their own teams of facts.

Experiment 234. Row and Column Cell Building Game with Cuisenaire Rods.

Let the children prepare the addition table and demonstrate the results with corresponding heights of rods.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8



You can give them other addition tables and they will find it interesting to construct structures of rods.

Experiment 235. Factors with Cuisenaire Rods.

The children form the following patterns

$\square$ Pattern for 1 $1 \times 1 = 1$ Factor of 1 is 1

$\begin{array}{|c|} \hline R \\ \hline W \\ \hline \end{array}$ Pattern for 2 $1 \times 2 = 2$ Factors of 2 are 1 and 2
 $2 \times 1 = 2$

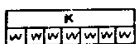
$\begin{array}{|c|} \hline 6 \\ \hline W \\ \hline \end{array}$ Pattern for 3 $1 \times 3 = 3$ Factors of 3 are 1 and 3
 $3 \times 1 = 3$

$\begin{array}{|c|} \hline P \\ \hline R \\ \hline W \\ \hline \end{array}$ Pattern for 4 $1 \times 4 = 4$ Factors of 4 are 1, 2 and 4
 $2 \times 2 = 4$
 $4 \times 1 = 4$

$\begin{array}{|c|} \hline Y \\ \hline W \\ \hline \end{array}$ Pattern for 5 $1 \times 5 = 5$ Factors of 5 are 1 and 5
 $5 \times 1 = 5$



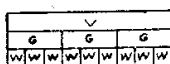
Pattern for 6 $1 \times 6 = 6$ Factors of 6 are 1, 2, 3
 $2 \times 3 = 6$ and 6
 $3 \times 2 = 6$
 $6 \times 1 = 6$



Pattern for 7 $1 \times 7 = 7$ Factors of 7 are 1 and 7
 $7 \times 1 = 7$



Pattern for 8 $1 \times 8 = 8$ Factors of 8 are 1, 2, 4
 $2 \times 4 = 8$ and 8
 $4 \times 2 = 8$
 $8 \times 1 = 8$



Pattern for 9 $1 \times 9 = 9$ Factors of 9 are 1, 3 and 9
 $3 \times 3 = 9$
 $9 \times 1 = 9$



Pattern for 10 $1 \times 10 = 10$ Factors of 10 are 1, 2, 5
 $2 \times 5 = 10$ and 10
 $5 \times 2 = 10$
 $10 \times 1 = 10$

The last pattern is interpreted as follows : 10 can be obtained by considering one rod of length 10 or 2 rods of length 5 or 5 rods of length 2 or 10 rods of length 1. The children can be encouraged to continue the pattern further.

Children can classify numbers as follows :

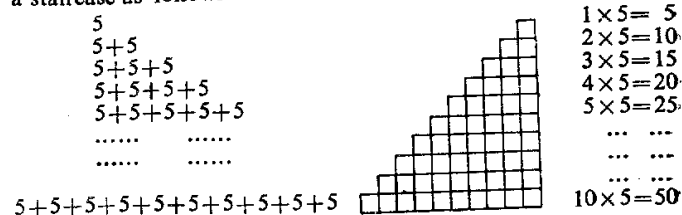
Just one factor : 1
 Just two factors : 1, 3, 5, 7
 More than 2 factors : 4, 6, 8, 9, 10

Numbers with just two factors are called **prime**, those with more than two factors are called **composite**. Number one is a special class by itself.

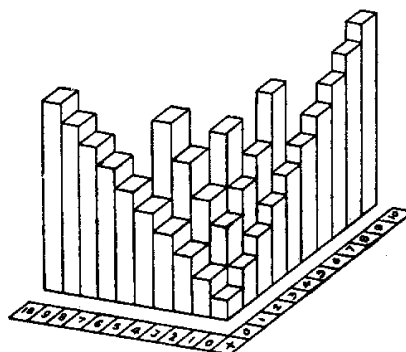
Experiment 236. Multiplication Tables with Cuisenaire Rods.

Children may prepare all the multiplication tables with Cuisenaire rods regarding multiplication as repeated addition. Thus to prepare multiplication table of 5, they have to find 1×5 , 2×5 , 3×5 ,, 10×5 i.e. they have to form trains of one yellow rod, 2 yellow rods, 3 yellow rods,, ten yellow rods and measure their lengths with the help of orange and yellow (or white) rods. They can arrange

a staircase as follows



After they have made the tables, they can form the following model with orange and white rods only.



Let the children also do some division sums as illustrated below :

To find $20 \div 5$, they place two orange rods to make 20 and then find how many yellow rods they can place alongside.

Alternatively they may take a train of twenty white rods and then go on removing five at a time and find the number of times they have to remove.

Experiments : 237 to 250

Experiment 237. Construction and Use of a Flannel Board.

This is an effective powerful teaching aid and is known by different names such as visual board, flick board, slap board, felt board, coherograph, videograph or flannel graph. For the preparation of this board, we require three items *viz.* backing material, covering material and fixing material. The backing material may be plywood, press board, masonite, wall board or heavy card board. The covering material may be flannel, suede, corduroy or khadi in any colour, though dark green, black or grey are preferable. The fixing material may be rubber, cement, tape, staples, nails, thumb tacks or drawing pins. The board may be in any convenient size or shape. The cloth is to be stretched so that no wrinkles are left.

Children may be encouraged to make their own flannel boards *e.g.* they may make these by covering with flannel or old blanket pieces or even khadi the top of rectangular biscuit boxes, in which they may keep their cut-outs.

Improvised flannel boards may be made by turning over a small table and spreading a khadi table cover over it or even by spreading a blanket over a vertical charpoy.

The figures or cut-outs may be made of any light weight and flat material, only the fibres should interlock. Some of the materials which are useful are : flannel, felt, balsa wood, steel wood, twill, blotting paper, velvet, sand paper, suede, corduroy, cellophane, emery paper etc.

The cut-outs may be in the form of animals, fruits, vegetables, trees, symbols, numbers, letters, words, geometrical figures etc.

The colourful cut-outs help in quickening the learning process, since children enjoy these and these create a visual impact on them. It helps to create situations involving motion. Beautiful patterns can be cut out once for all and used again and again.

Almost all the experiments in this book and in the preceding volume can be illustrated on the flannel board and it will be useful to give the children flannel or khadi or sand paper or even ordinary paper and a pair of scissors and let them try to illustrate all the ideas on their own small flannel boards.

Experiment 238. Cuisenaire Rods and Flannel Board.

Instead of rods, the children can be given sets of pieces of flannel of lengths 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 in the same colours as the Cuisenaire rods and they can have same free play activities, form trains, learn all addition, subtraction, multiplication and division facts, illustrate commutative and associative laws, study partitions, study tens and units, play games, form staircases as in experiments in chapter thirteen.

Experiment 239. The Hundred Chart.

The hundred numbers may be printed on 10 flannel strips each containing 10 circles and these strips may be spread one after the other to give the complete hundred chart.

Different children may be called to the board and asked to place red circular cut-outs :

- (i) On all multiples of 2, all multiples of 3, all multiples of 4,all multiples of 10, all multiples of 11 etc.
- (ii) On all prime numbers.
- (iii) On all square numbers.
- (iv) On all perfect cubes.
- (v) On square numbers which are double or three times or four times of some other square numbers, whenever these exist.

They may also note the emerging patterns in all cases.

Experiment 240. Place Value.

We use flannel discs of different colours, one colour (say red) representing hundreds, a second colour (say blue) representing tens and a third (say white) representing units. The children may represent numbers like 342, 302, 320 with the help of these discs. They may also exchange one red disc for ten blue ones and one blue one for 10 white ones and may learn carrying over and borrowing for addition and subtraction respectively.

Experiment 241. Fundamental Operations.

A flannel tree may be put on the board. Some birds may be placed on it and some more may be shown joining them to give the concept of addition. For subtraction, some fruits may be shown growing on the tree and some of these may be shown to have fallen down. For multiplication, equal groups of birds or animals may be shown to be coming together. For division, a big group may be shown breaking up into smaller groups of equal size. The commutative law for addition can be illustrated by first placing three apples on the board and then two next or by placing two apples on the board first and then placing three apples next. The associative law for addition can be illustrated by showing three groups of 4, 3, 2 birds on the board and first bringing the second group near to first to get 7 and then bringing the last group near the first two to get 9 and secondly bringing the last two groups together first to get 5 and then bringing this combined group near to the first to get 9. The commutative law for multiplication can be illustrated by bringing two groups of three birds together and then bringing three groups of two birds together. The distributive law may be illustrated with such diagrams as,

$$\begin{array}{r} 0000 \\ 0000 \\ 0000 \end{array} \left| \begin{array}{r} 00 \\ 00 \\ 00 \end{array} \right. \begin{array}{r} 0000 \\ 0000 \\ 0000 \end{array} \begin{array}{r} 00 \\ 00 \\ 00 \end{array}$$

showing $3 \times 6 = 3 \times 4 + 3 \times 2$.

Distributive law for division over addition can be illustrated by

$$\begin{array}{r} 00000000 \\ 00000000 \\ \hline 00000000 \\ 00000000 \\ \hline 00000000 \\ 00000000 \end{array} \begin{array}{r} 0000 \\ 0000 \\ \hline 0000 \\ 0000 \\ \hline 0000 \\ 0000 \end{array} \begin{array}{r} 000 \\ 000 \\ \hline 000 \\ 000 \\ \hline 000 \\ 000 \end{array} \begin{array}{r} 00 \\ 00 \\ \hline 00 \\ 00 \\ \hline 00 \\ 00 \end{array}$$

showing $\frac{5}{3} \div \frac{2}{3} = \frac{5}{3} \div \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$

Similar figures may be used to show that commutative and associative laws do not hold for subtraction and division.

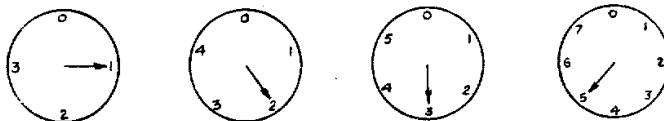
Experiment 242. Flannel Clocks.

A flannel clock is a circular piece of flannel cloth on which numbers 1 to 12 are printed as in ordinary clock dials and in addition two moving hands are provided.



The children can be taught to read time with this clock. Different clocks can be put on the board to show the times in different parts of the world at one given time.

Flannel clocks of the following types can be used to teach finite arithmetic or clock arithmetic.



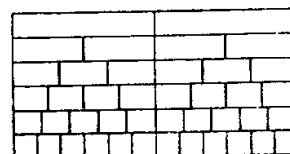
Similar clocks can be used to teach fractions and angles.

Experiment 243. Fractions.

Circular or rectangular pieces of flannel may be taken and by folding, the concept of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$ etc. may be illustrated.

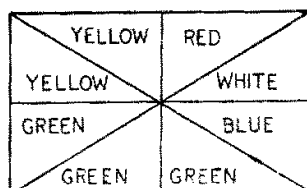
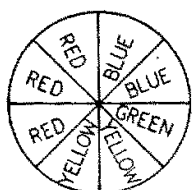
We may use cut-outs of the following shapes to show that :

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12} \dots \dots \dots$$

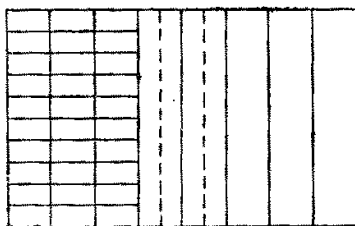


and then introduce the important idea of equivalent fractions.

We may use cut-outs of the following shapes to illustrate $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, etc.



We may use cut-outs of the following shape to show that $\frac{3}{6} + \frac{3}{6} = \frac{6}{6}$ and $\frac{5}{6} - \frac{2}{6} = \frac{3}{6}$ and to show that to add or subtract fractions with same denominator, we add or subtract numerators



Similarly to find $\frac{1}{2} + \frac{1}{3}$ or $\frac{1}{2} - \frac{1}{3}$, we take three equal rectangular cut-outs and divide each into six equal parts. We take half of the first (i.e. take three parts out of six), one-third from the second (i.e. take two parts out of six) and place them side by side or superpose them to show that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Experiment 244. Recognition of Coins, Numerals etc.

A flannel graph set showing shapes of all current coins, currency notes etc. can be made and these can be used to simulate buying and selling situations.

Another flannel graph set may show some important foreign coins and their comparative values as against Indian coins.

Another flannel graph set may show Roman, Greek, Egyptian and Hindu-Arabic numerals and children may be asked to match these on the board

Experiment 245. Mathematical Vocabulary.

Two boxes are given to the children. One contains mathematical terms and the other contains illustrations of these. The children are required to match these. They first put these in first and third columns of the flannel board and then take out pieces from the third column and place these in the second column in the proper places.

Commutative law for addition	
Commutative law for multiplication	$4 \times (3 + 2) = 4 \times 3 + 4 \times 2$
Associative law for addition	$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}$
Associative law for multiplication	
Distributive law	$3 + 4 = 4 + 3$
Rectangle	4, 9, 16, 25, 36, 49, 64,
Hexagon	
Octagon	$3 \times 5 = 5 \times 3$
Equivalent fractions	2, 3, 5, 7, 11, 13, 17, 19, 23, ...
Prime numbers	$(2 + 3) + 4 = 2 + (3 + 4)$
Square numbers	1, 3, 6, 10, 15, 21,
Triangular numbers	$(2 \times 3) \times 4 = 2 \times (3 \times 4)$

Experiment 246. Recognition of Geometrical Shapes.

A large number of three-sided, four-sided, five-sided, six-sided, seven-sided, eight-sided flannel cut-outs may be flicked on the board. Children may be given one triangular piece and they may be asked to choose all similar shapes on the board. Most probably they would choose triangular pieces. In this way children may learn that number of sides may be an important factor for classification.

Children may also distinguish between regular and non-regular polygons, between convex and non-convex polygons etc.

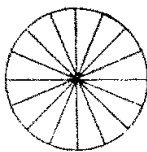
Quadrilaterals of different shapes may be put on the table and let the children learn to classify them into rectangles, squares, rhombuses, trapezia, kites etc.

Experiment 247. Demonstration of Geometrical Results. Area of a Circular Region.

By just folding of flannel cut-outs on the board, results like the following can be demonstrated :

A parallelogram is divided into two equal halves by a diagonal, the opposite sides of parallelogram are equal, every diameter bisects a circle, radius of a circle is one-half the diameter, opposite angles of a parallelogram are equal, the line joining the middle points of the sides of a triangle is one half the base.

A circular piece may be cut into a number of equal sections. These pieces may be later fitted into the circular region as well as into a rectangular region as follows :



and let the children deduce that

Area of a circular region = half the circumference $\times$ radius.

Experiment 248. Tessellations.

Experiments on tessellations can be easily done on a flannel board e.g. children be given equal triangular pieces or equal quadrilateral pieces or equal hexagonal pieces and may be asked to cover the flannel boards with these. They may also see that they cannot cover up the board with regular pentagonal pieces or with equal circular pieces.

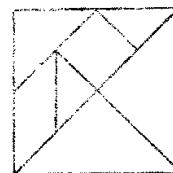
They may be given equilateral triangular pieces of the same size and be asked to make equilateral triangular regions of large size. They require 4, 9, 16, pieces and construct triangular regions of double, triple, four times the size etc. Similar experiments can be done with squares, parallelograms, trapezia etc.



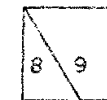
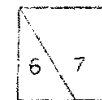
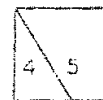
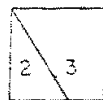
Experiment 249. Dissection Problems.

Children may take any figure, cut it into any number of pieces, mix these up and then try to reassemble these to get the original shape or even a new shape e.g. they may reassemble the following 7 pieces to get the square.

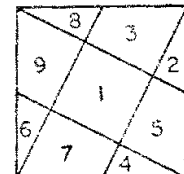
They may be asked to reassemble the dissected parts of the three hexagons to get the large hexagon.



Similarly they may be asked to show that the following dissected parts may be reassembled to get a larger square.

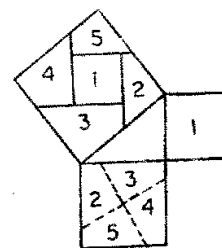


These dissections may be used to find the side of new hexagon or of new squares. Dissections may also be used to show that the areas of two figures are equal, even though the two figures may look different.

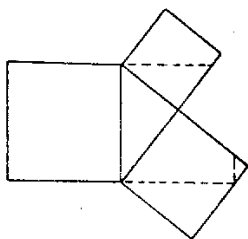


Experiment 250. Verification of Pythagoras Theorem by Dissection.

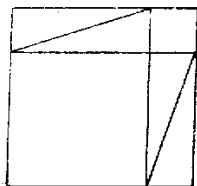
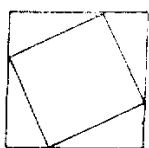
- (a) Take a right angled triangle and draw squares on all the three sides. Find the centre of the larger of the two sides by finding the point of intersection of diagonals. Through this centre draw one line parallel to the diagonal and one line perpendicular to the diagonal. The children can then show by actually placing the square 1 and the four pieces 2, 3, 4, 5 to fit exactly the square on the diagonal.



- (b) Draw a line perpendicular to the diagonal in the smaller square and a line parallel and perpendicular due to diagonal in the square on the bigger side. These five pieces should fit exactly the square on the diagonal.



- (c) Place five congruent right-angled triangles inside a square (each edge of which is the sum of the two sides of the right-angled triangles) in two different ways as shown below :



and let the children see how this again verifies Pythagoras theorem.

Experiments : 251 to 262

Experiment 251. Additive and Subtractive Lattices.

Start with any number, say 1 ; add a given number say 2, when moving towards the right and subtract the same number when moving towards the left. Add a given number, say 3, when going upwards and subtract the same number when going down.

These steps can be repeated indefinitely, to give

	:	:	:	:	:	:	:
...	12	14	16	18	20	22	24 ...
...	9	11	13	15	17	19	21 ...
...	6	8	10	12	14	16	18 ...
...	3	5	7	9	11	13	15 ...
...	0	2	4	6	8	10	12 ...
...	-3	-1	1	3	5	7	9 ...
...	-6	-4	-2	0	2	4	6 ...
...	-9	-7	-5	-3	-1	1	3 ...
...	-12	-10	-8	-6	-4	-2	0 ...
	:	:	:	:	:	:	:

This lattice may be denoted by $(1 \xrightarrow{+2}, \uparrow +3)$. It is the same as $(3, \xleftarrow{-2}, \downarrow -3)$, but it is different from $(2, \xrightarrow{+2}, \uparrow +3)$. There are essentially two different lattices with rules $\xrightarrow{+2}, \uparrow +3$. How many different lattices are there with rules $\xrightarrow{+3}, \uparrow +2$? The number of different lattices with rules $\xrightarrow{+a}, \uparrow +b$ is obviously the smaller of the natural numbers a and b . How many different lattices are there with rules $\xrightarrow{+a}, \uparrow -3$? How many different lattices are there with rule $\xrightarrow{+a}, \uparrow b$, where a and b are integers? The number of lattices is the smaller of $|a|, |b|$. Children will get a lot of practice in addition and subtraction with such lattices.

In the above lattice, we write $1 \rightarrow = 3, 1 \uparrow = 4, 1 \nearrow = 6, 1 \nwarrow = -1, 1 \downarrow = -2, 1 \swarrow = -4$. We have thus 6 functions defined. $\rightarrow$ means 'add 2', $\nwarrow$ means subtract 2, $\nearrow$ means add 5, $\swarrow$ means subtract 5 and so on.

$$1 \rightarrow \nearrow = 3 \uparrow \nearrow = 6 \nearrow = 11.$$

These are examples of functions of functions. It is obvious that the order of the operations is immaterial. We verify

$$1 \nearrow \rightarrow \uparrow = 6 \rightarrow \uparrow = 8 \uparrow = 11.$$

Children may find it interesting to compute expansions like

$$5 \rightarrow \uparrow \rightarrow \downarrow \nearrow \downarrow \nwarrow \nwarrow \uparrow \rightarrow.$$

They may note that $\rightarrow, \nwarrow; \uparrow, \downarrow; \nearrow, \swarrow$ cancel each other, that $\nearrow$ is equivalent to $\rightarrow \uparrow$, $\swarrow$ is equivalent to $\nwarrow \downarrow$ and so on. They may calculate how many steps are to be moved towards the right, how many are to be moved towards the left, how many are to be moved upwards and how many are to be moved downwards etc. and then they can find easily the image or map of any number.

Experiment 252. Multiplication and Division Lattices.

Here $\xrightarrow{+a}$ may mean multiply by a and $\uparrow b$ may mean multiply by b , then $\xleftarrow{-a}$ means divide by a , $\downarrow b$ means divide by b and a and b can be any non-zero natural numbers. What do we get if we start with 0? On the next page we give the lattices $1, \xrightarrow{\times 2}, \uparrow \times 3$.

	:	:	:	:	:	:	:	:	:	:
...	$\frac{81}{16}$	$\frac{81}{8}$	$\frac{81}{4}$	$\frac{81}{2}$	81	162	324	648	1296	2592 ...
...	$\frac{27}{16}$	$\frac{27}{8}$	$\frac{27}{4}$	$\frac{27}{2}$	27	54	108	216	432	864 ...
...	$\frac{9}{16}$	$\frac{9}{8}$	$\frac{9}{4}$	$\frac{9}{2}$	9	18	36	72	144	288 ...
...	$\frac{3}{16}$	$\frac{3}{8}$	$\frac{3}{4}$	$\frac{3}{2}$	3	6	12	24	48	96 ...
...	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32 ...
...	$\frac{1}{48}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{8}{3}$	$\frac{16}{3}$	$\frac{32}{3}$...
...	$\frac{1}{144}$	$\frac{1}{72}$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{9}$	$\frac{16}{9}$	$\frac{32}{9}$...
...	$\frac{1}{432}$	$\frac{1}{216}$	$\frac{1}{108}$	$\frac{1}{54}$	$\frac{1}{27}$	$\frac{2}{27}$	$\frac{4}{27}$	$\frac{8}{27}$	$\frac{16}{27}$	$\frac{32}{27}$...
...	.	.	.	.	.	.	.	.	.	.
...	.	.	.	.	.	.	.	.	.	.

Since in these lattices, we are considering all rational numbers, even with $\xrightarrow{\times 2}, \uparrow \times 3$, we can get an infinity of lattices. $\xrightarrow{\times a}, \uparrow \times b$ gives the same result as $\nearrow a \times b$.

The children will get a great deal of practice of multiplication and division by forming such lattices. Again here $(\rightarrow, \nwarrow), (\uparrow, \downarrow), (\nearrow, \swarrow)$ cancel out in pairs and the result of a number of successive operations can be easily found e.g. in the above table

$$1 \nearrow \rightarrow = 6 \rightarrow \uparrow = 12 \uparrow = 36.$$

Alternatively $\nearrow \rightarrow \uparrow$ means 2 steps towards the right (multiply by 4) and 2 steps upwards (multiply by 9).

All numbers in the above lattice will be of the form $2^m \times 3^n$, where m, n are positive or negative integers or zero. Conversely no number which is not of this form can occur in the lattice. Given any number in the lattice we can find where it is, for we shall express it in the form $1 \times 2^m \times 3^n$ and then starting with 1, we shall move m steps towards the right or left according as m is positive or negative and n steps upwards or downwards according as n is positive or negative. The shortest distance to this point will be of $|m| + |n|$ steps.

In the lattice $1, \rightarrow, \uparrow \times 4$, all numbers are of the form $2^m \times 4^n$ or of the form 2^{m+2n} . The children may like to find the lengths of the shortest paths to this point for different values of m and n .

Experiment 253. Possibilities of Other Lattices.

Let us try to form a lattice $a, \rightarrow, \uparrow \times c$.
 In this case $a \rightarrow \uparrow = (a+b) \uparrow = (a+b) \times c$
 and $a \uparrow \rightarrow = ac \rightarrow = ac+b$.

These two will in general be different unless $c=1$. In this case, however

$$\uparrow \times 1 = \uparrow + 0,$$

so that both motions towards the right and upwards are additive. The lattice where $\rightarrow$ is additive (multiplicative) and $\uparrow$ is multiplicative (additive) do not exist. Children can reach this conclusion after trying more numerical examples.

Experiment 254. Three-Dimensional Lattices.

In this case $d, \rightarrow, \uparrow + b, \nearrow + c$ will mean that the d will have the image $d+ma+nb+pc$ after m steps in one direction, n steps in a perpendicular direction and p steps in a direction perpendicular to these two directions. Lattices where all the three operations are additive (i.e. consist in addition of positive or negative integers) or multiplicative (i.e. consists of multiplication by positive or negative numbers) are possible. However, lattices where some operations are additive and others are multiplicative are not possible. The children can easily see this result by working out a few special cases.

Experiment 255. Non-Rectangular Lattices.

Consider the lattice.

::	::	::	::	::
11	12	13	14	15
7	8	9	10	
4	5	6		
2	3			
1				

The problem may be to find where does the number 10,000 lie in this lattice. The children may note that the diagonal is formed by the triangular numbers 1, 3, 6, 10, 15,.....which are the sums

of the first one, two, three, four, five.....natural numbers so that the last number of the n th row is $n \frac{(n+1)}{2}$

$$\text{Put } n \frac{(n+1)}{2} = 10,000$$

$$\text{or } n(n+1) = 20,000$$

n and $n+1$ are nearly equal when n is large. Therefore n is approximately 141, since

$$\frac{141 \times 142}{2} = 10,011$$

This is the last number of the 141st row, but the row has 141 elements. If we count from the end, 10,000 will be twelfth number and from the beginning it will be 130th number. Thus 10,000 is 130th number of 141st row.

A more difficult problem occurs in the case of lattices of odd or even numbers only viz.

..	..	..	..	..	..	..	..	..	..
22	24	26	28	30	..	..	..	..	..
14	16	18	20		21	23	25	27	29
8	10	12			13	15	17	19	
4	6				or	7	9	11	
2						3	5		
							1		

In the first case, the last number of n th row is given by $n(n+1)$ and in the second case the last number of n th row is given by $n(n+1)-1$.

If we want 10,000 in the first case, we get

$$n(n+1) = 10,000$$

Trying $n=100$, we get $n(n+1)=10100$, which shows that 10,000 is the first number of the 100th row. We can proceed similarly in the second case. Children may like to investigate other lattices of the same type.

Experiment 256. Loci on Lattices.

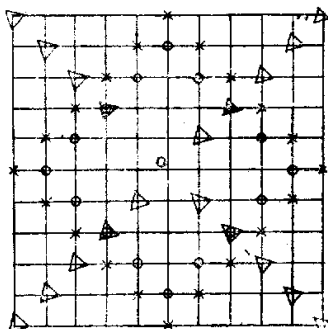
Consider a square grid, say on a graph paper. Take any intersection O as a fixed point. From this we can move any number of steps towards the right or left, up or down. Let the children find the points, the horizontal distance of each of which from O is 3 steps and vertical distance is 2 steps.

Let the children now find all points on the lattice whose sum of horizontal and vertical distances from O is 5. These points are shown in the figure by crosses. These lie on four straight line segments.

Similarly those points whose sum of horizontal and vertical distances is 4 are shown by circles. They also lie on four line segments. Those whose horizontal and vertical distances are equal are shown by Δ 's. They all lie on two straight lines.

The children may like to find the following points and the straight lines or curves connecting them :

- (i) all those points whose horizontal distance from O is 2 (vertical distance may be anything)
- (ii) all those points whose vertical distance from O is 3 (horizontal distance may be anything)
- (iii) all those points whose horizontal distance is double of the vertical distance.
- (iv) all those points whose vertical distance is double of the horizontal distance.
- (v) all those points for which the difference between horizontal and vertical distances is 1.
- (vi) all those points the sum of squares of whose horizontal and vertical distances is 25
- (vii) all those points for which squares of vertical distances are equal to vertical distances.
- (viii) all those points for which squares of vertical distances are equal to horizontal distances.



The children can specify any relation between horizontal and vertical distances and find the corresponding points.

Experiment 257. Introducing Co-ordinates.

The position of any point on the grid is known if we know its horizontal and vertical distances from O and we know further whether the horizontal distance is towards the right or left and whether the vertical distance is upwards or downwards. The horizontal distances towards the right are considered positive, those towards the left are considered negative. Similarly the vertical distances upwards are considered positive and those downwards are considered as negative. The point P is denoted by (3, 2) showing that it is 3 horizontal steps towards the right and 2 vertical steps upwards. Similarly the point denoted by (-3, -2) is 3 horizontal steps towards the left and 2 vertical steps downwards.

We shall denote the horizontal coordinate by x and the horizontal distance by $|x|$. The horizontal coordinate may be positive or negative, but the horizontal distance is always positive. Similarly we shall denote the vertical coordinate by y and the vertical distance by $|y|$.

The above loci are then represented by the equations

$$\begin{aligned}
 |x| + |y| &= 5 \\
 |x| + |y| &= 4 \\
 |x| &= |y| \\
 |x| &= 2 \\
 |y| &= 3 \\
 |x| &= 2|y| \\
 |y| &= 2|x| \\
 |x| &\sim |y| = 1 \\
 |x|^2 + |y|^2 &= 25 \\
 |x|^2 &= |y| \\
 |y|^2 &= |x|
 \end{aligned}$$

Experiment 258. Other Loci on Lattices.

We are considering points with integral coordinates only. $|x| + |y| = 5$ is satisfied by :

(0, 5), (1, 4), (2, 3), (3, 2), (4, 1), (5, 0); (4, -1), (3, -2), (2, -3), (1, -4), (0, -5);
 (-1, -4), (-2, -3), (-3, -2), (-4, -1), (5, 0); (-4, 1), (-3, 2), (-2, 3), (-1, 4).

How many points satisfy $x+y=5$?

The children may now try to draw the loci $x+y=5$, $x-y=5$, $x=y$, $x=2$, $y=3$, $x=2y$, $y=2x$, $x-y=1$, $x^2+y^2=25$, $x^2=y$, $y^2=x$ and compare these with what they got in the last section. The first 8 loci are straight lines, the ninth is a circle and tenth and eleventh are called parabolas.

Experiment 259. Ellipses with Integral Co-Ordinates.

Those teachers who want their children to draw ellipses, while they can use integral coordinates only, will find the following example useful:

$$x^2+3y^2=4(a^2+ab+b^2)$$

It is satisfied by the following 12 points:

$$\{\pm(a-b), \pm(a+b)\}, \{\pm(2b+a), \pm a\}, \{\pm(2a+b), \pm b\}$$

Here a and b are any positive integers whatsoever. Thus $a=1$, $b=2$ gives $x^2+3y^2=28$ which is satisfied by

$(1, 3)$, $(-1, -3)$, $(1, -3)$, $(-1, 3)$, $(5, 1)$, $(5, -1)$, $(-5, 1)$, $(-5, -1)$, $(4, 2)$, $(4, -2)$, $(-4, 2)$, $(-4, -2)$.

Similarly the following 12 points lie on ellipse

$$x^2-xy+y^2=a^2+ab+b^2:$$

$$(a, -b), (-a, b), (b, -a), (-b, a), (a+b, b), (-b, -b-a), (b, b-a), (-b-a, -b), (a+b, a), (-a-b, -a), (a, a+b), (-a, -a-b).$$

The children can draw a large number of ellipses by choosing different positive integral values, for a and b .

Experiment 260. Shortest Paths on Lattices.

Let the children find the possible paths between a fixed point O and any point P with coordinates x, y and find their lengths i.e. total number of steps to go from O to P.

In the figure below we show some paths from O to P (2, 3). We indicate the paths and give their lengths in brackets.

N	M	P	L		
H	I	J	K		
G	F	E	D		
O	A	B	C		

OABEJP (5), OAFEJP (5), OGFEJP (5)

OGFIJP (5), OABEFIJP (7), OAFEJIMP (7),

OABCDEJKLP (9)

Can they get a path shorter than 5?

Can they get a path of length 6 or 8?

How many different paths of length 5 can they find?

Do all paths lying in the rectangle OBPN give the shortest length? Can they show that the shortest path to (x, y) has length $|x| + |y|$? Can they find paths of length $|x| + |y| + 1$ or of $|x| + |y| + 3$ or of $|x| + |y| + n$ where n is an odd number?

Which are those points for which there is only one shortest path?

Which are those points for which there are only two shortest paths?

Which are those points for which there are only three shortest paths?

Experiment 261. Shortest Paths on Three-Dimensional Lattices.

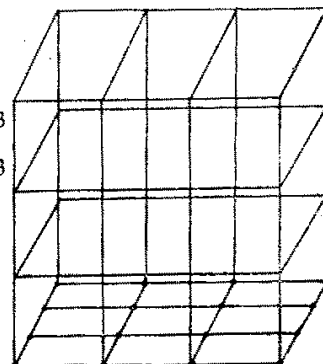
A three-dimensional wire lattice of size $6 \times 6 \times 6$ or $8 \times 8 \times 8$ can be used to investigate problems similar to those given above. If possible, each child may be given a wire lattice with 64 lattice points and children may find shortest paths between one fixed point and other points of the lattice. Different paths may be shown by threads of different colours. It may be shown that the shortest length to the point (x, y, z) is $|x| + |y| + |z|$ and that we cannot have a path of length $|x| + |y| + |z| + n$, where n is odd.

The children may also point out all points for which

$$|x| + |y| + |z| = 3$$

$$\text{or } |x| + |y| + |z| < 3$$

$$\text{or } |x|^2 + |y|^2 = 25$$



Consider the following lattice. Let the children try to guess the pattern.

1	1	1	1	1	1	1	
2	4	8	6	2	4	8	
3	9	7	1	3	9	7	
4	6	4	6	4	6	4	
5	5	5	5	5	5	5	
6	6	6	6	6	6	6	
7	9	3	1	7	9	3	
8	4	2	6	8	4	2	
9	1	9	1	9	1	9	

The first column gives all the numbers 1 to 9. The second column gives the unit digits of their squares. The remaining successive columns give the unit digits of their cubes, fourth powers, fifth powers etc. The pattern shows that

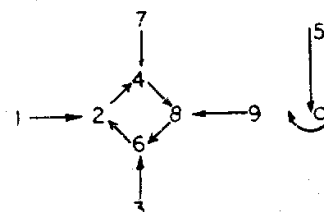
- (i) first, fifth, ninth, thirteenth.....powers of any number have the same unit digit.
- (ii) second, sixth, tenth, fourteenth powers of any number have the same unit digit.
- (iii) third, seventh, eleventh, fifteenth powers of any number have the same unit digit.
- (iv) fourth, eighth, twelfth and sixteenth powers of any number have the same unit digit.

The children can answer questions like the following :

What is the unit digit in 9^{100} , 7^{256} , 5^{999} , 3^{425} etc. ? Again consider the lattice :

1	2	4	8	6	2	4	
2	4	8	6	2	4	8	
3	6	2	4	8	6	2	
4	8	6	2	4	8	6	
5	0	0	0	0	0	0	
6	2	4	8	6	2	4	
7	4	8	6	2	4	8	
8	6	2	4	8	6	2	
9	8	6	2	4	8	6	

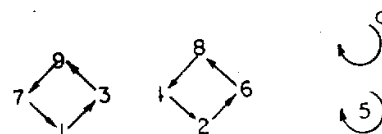
Successive columns give the unit digits of the numbers obtained by successively doubling numbers in the first column. The pattern that emerges is given by the following diagram :



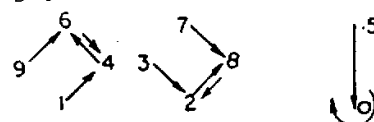
The last figure shows that in doubling 5, we get unit digit 0 and then in successive doubling 0 continues as 0.

Similarly in successive multiplying by 3, 4, 5, 6, 7, 8, 9 the children get the patterns :

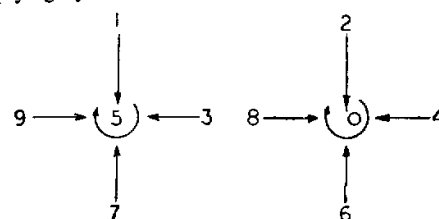
Trebling :



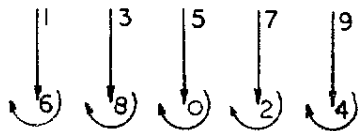
Multiplying by 4 :



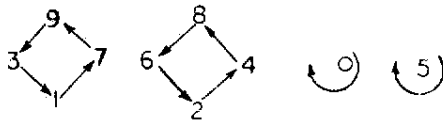
Multiplying by 5 :



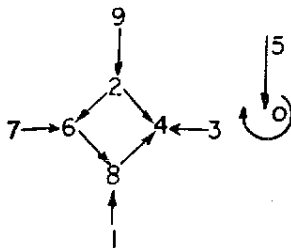
Multiplying by 6 :



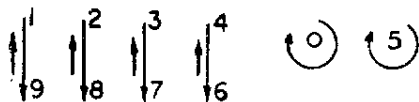
Multiplying by 7 :



Multiplying by 8 :



Multiplying by 9 :



The children may compare the patterns for

- (i) Multiplying by 3 and 7
- (ii) Multiplying by 2 and 8
- (iii) Multiplying by 2, 3 and 6.

Experiments : 263 to 272

Experiment 263. Patterns in addition of columns.

Write all natural numbers in five columns as follows :

A	B	C	D	E
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
...	...	...	...	...
...	...	...	...	...

Let us call these columns as A, B, C, D, E. Let the children add *any* number in column A to *any* number in column B. To which column does the sum belong? Whatever numbers children choose from columns A and B, the sum always belongs to column C. We express this by writing

$$A + B = C$$

Similarly any two columns can be added and the sum is always some column. The addition table for columns comes out to be

+	A	B	C	D	E
A	B	C	D	E	A
B	C	D	E	A	B
C	D	E	A	B	C
D	E	A	B	C	D
E	A	B	C	D	E

Let the children note the following patterns :

- (i) The sum of any two columns is always a column i.e., if we choose one number from each of two columns, the sum always belongs to the same column. We have a set of five columns and the sum of any two columns of the set is a column belonging to this set. The set is said to be *closed* with respect to the operation of addition of columns.
- (ii) Let the children find $(A+B)+C$ and $A+(B+C)$. Do they expect these to be equal? Remember we are adding columns and not numbers. Yet we find

$$(A+B)+C=C+C=A$$

$$A+(B+C)=A+E=A$$

The children can choose any three columns, add second to first and then add the third to the sum of the first two. Next let them add the sum of third and second to the first. They will always find these equal. The first choice can be any one of the five columns, the second choice can be any one of the five columns, the third choice can also be any one of the five columns. There can be thus 125 triplets like AAA, AAB, AAC, AAD, AAE, ABC, BCD, CDE, CBA, etc. The children can verify the associative law for as many of the triplets as they like or they may do this collectively. Twenty-five pairs AA, AB, AC, AD, AE, BA, BB, BC, BD, BE, CA, CB, CC, CD, CE, DA, DB, DC, DD, DE, EA, EB, EC, ED, EE can be formed. By adding A, B, C, D, E we can form five triplets from each pair. Each child can be given a pair and let him verify the associative law for the five triplets which start with his pair.

- (iii) From the table, it appears that

$$\begin{aligned} A + E &= E + A = A \\ B + E &= E + B = B \\ C + E &= E + C = C \\ D + E &= E + D = D \\ E + E &= E + E = E \end{aligned}$$

The column E has therefore a special property. What is it? When it is added to any column or any column is added to it, we get that column. This column E will be called the *additive identity*. Its behaviour is the same as that of the number 0 in the number system. Thus we find that in our set of five columns, there is a column E which is the additive identity.

- (iv) In the system of integers there is a number 0 and for every integer a there is a number $-a$ such that

$$a + (-a) = 0 = (-a) + a.$$

$(-a)$ is called the *additive inverse* of a .

In our set of columns, does every column have an additive inverse i.e., corresponding to any column, does there exist another column such that the sum of the two columns is the column E, which is the additive identity. We find :

$A + D = E$, $B + C = E$, $C + B = E$, $D + A = E$, $E + E = E$, so that A and D are additive inverses of each other, B and C are additive inverses of each other and E is its own additive inverse.

- (v) Finally it is easy to see that the commutative law for addition holds viz.,

$$\begin{aligned} A + B &= B + A, A + C = C + A, A + D = D + A, A + E = E + A, \\ B + C &= C + B, B + D = D + B, B + E = E + B, C + D = D + C, \\ C + E &= E + C, D + E = E + D \end{aligned}$$

Let the children verify all these.

Let the children take 4 or 6 or 7 columns as follows :

A	B	C	D	E	F	G
1	2	3	4	5	6	7
5	6	7	8	9	10	11
9	10	11	12	13	14	15
13	14	15	16	17	18	19
17	18	19	20	21	22	23
21	22	23	24	25	26	27
25	26	27	28	29	30	31
29	30	31	32	33	34	35
...	...	...	...	...	...	...

and verify in each case the five properties given above.

In the first case D acts as the additive identity, A and C are additive inverses, and B is its own inverse. In the second case F acts as the additive identity, A and E, B and D and C and C are additively inverse pairs, while in the third case G acts as the additive identity, A and F, B and E and C and D are additively inverse pairs.

Experiment 264. Clock Arithmetic (Additive).

In this case we start with the numbers and proceed as above and instead of indicating columns by A, B, C, D and E, we indicate them by 0, 1, 2, 3 and 4.

	0	1	2	3	4
0	0	1	2	3	4
5	5	6	7	8	9
10	10	11	12	13	14
15	15	16	17	18	19
...	...	...	...	...	...

We may note that 0, 1, 2, 3, 4 are not numbers but indicate columns of numbers or, so we sometimes prefer to call them, classes of numbers.

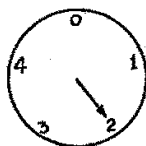
The addition table is as follows :

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

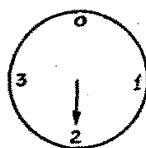
We can again verify all the five properties above. 0 acts as the additive identity, 1 and 4 are additive inverses and so are 2 and 3. It may also be noted that we will get the same addition table if we follow the rule :

Add two classes as if they were numbers ; if this sum is less than 5, this gives the sum class, if it is 5 or more, subtract 5.

The same arithmetic is obtained from the following clock :



The children can find similar clock arithmetics for the following clocks and verify rules similar to the one given on the previous page.



Experiment 265. Additive Group Structure with Cuisenaire Rods.

In each of these above examples, we have a set of elements, and there is an operation defined there called addition such that

- sum of any two elements of the set is an element of the set.
- associative law for addition holds.
- there exists an additive identity i.e. an element of the set which when added to any element of the set gives that element of the set.
- for every element there exists an additive inverse such that the sum of the element and the additive inverse is the additive identity.
- the commutative law for addition holds.

A set of elements with an operation of addition satisfying the above five properties is called a *commutative additive group* or an *Abelian additive group*. (Abel was the name of a famous mathematician who lived in the years 1802—1829 and in spite of death at a very early age he did very good work in mathematics) If only the first four properties are satisfied, we simply call it an *additive group*.

Let the children verify the five properties explicitly in each case and let them give examples of additive groups with 8 or 9 or 10 elements.

They may take the Cuisenaire rods, write numbers 1, 11, 21, 31, 41, 51 on the white rod, numbers 2, 12, 32, 42, 52, on the red rod

and so on and then form an addition table as follows :

+	W	R	G	P	Y	D	K	N	U	O
W	R	G	P	Y	D	K	N	U	O	W
R	G	P	Y	D	K	N	U	O	W	R
G	P	Y	D	K	N	U	O	W	R	G
P	Y	D	K	N	U	O	W	R	G	P
Y	D	K	N	U	O	W	R	G	P	Y
D	K	N	U	O	W	R	G	P	Y	D
K	N	U	O	W	R	G	P	Y	D	K
N	U	O	W	R	G	P	Y	D	K	N
U	O	W	R	G	P	Y	D	K	N	U
O	W	R	G	P	Y	D	K	N	U	O

W=White (1), R=Red (2), G=Green (3), P=Purple (4), Y=Yellow (5), D=Dark Green (6), K=Black (7), N=Brown (8), U=Blue (9), O=Orange (10).

Since three columns viz. black, brown and blue start with same letter B, we use their last letters K, N and U to denote them. In the last case we prefer U to E as it gives a sound nearer to blue.

Here O is the additive identity, and (W, U), (G, K), (P, D), and (Y, Y) are additively inverse pairs.

Experiment 266. Additive Group of Integers.

The set $\{1, 2, 3, 4, 5, \dots\}$ is called the set of Natural Numbers and is denoted by N.

The set $\{0, 1, 2, 3, 4, 5, \dots\}$ is called the set of whole numbers and is denoted by W.

The set $\{\dots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \dots\}$ is called the set of integers and is denoted by I.

There is an addition operation defined in all these three sets. Let the children see whether all the five properties are satisfied.

	N	W	I
(i) Is the sum of two members of the set a member of the set ?	Yes	Yes	Yes
(ii) Does the associative law hold ?	Yes	Yes	Yes
(iii) Does there exist an additive identity ?	No	Yes	Yes
(iv) Does every element have an additive inverse ?	No	No	Yes
(v) Does the commutative law hold ?	Yes	Yes	Yes.

Thus the set of natural numbers is not a group for addition, the set of whole numbers is not a group for addition while, the set of integers is a group for addition.

We start with the set of natural numbers. It satisfies the first two properties, but does not satisfy the third. We define the number zero and form the set of whole numbers so that it may satisfy the third property also. The set W does not still satisfy the fourth property and so we introduce negative integers. The set I forms an additive group and is a satisfactory structure from the point of view of the operation of addition.

This explains why zero and negative integers had to be 'invented' in mathematics.

As we shall see later, the set I is not satisfactory from the point of view of the operation of multiplication and this will show the necessity of the 'invention' of rational numbers.

We have talked of the word 'structure'. This is a powerful word. Just as a building has not only bricks, cement and mortar, it has a structure ; similarly numbers in mathematics are not isolated entities, they are connected by laws of commutativity associativity, existence of additive identity, existence of inverses etc., and it is these which give a structure to the number system and it is due to the resulting structure that number systems are useful in mathematics and other sciences.

The children may now try to show that the positive fractions do not form an additive group (why ?)

Experiment 267. Additive Group of Triplets.

Consider the triplet $(\square, \triangle, \diamond)$ where the number in any frame can be 0 or 1. There are eight such triplets viz.

$(0,0,0), (1,0,0), (0,1,0), (0,0,1), (0,1,1), (1,0,1), (1,1,0), (1,1,1).$

We define sums of two triplets as that triplet obtained by adding corresponding components provided further that if the sum of two components is 2, we write 0, i.e., we use

$$0+0=0, 0+1, 1+0=1, 1+1=0$$

The children can now form the addition table :

+	(0,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(0,1,1)	(1,0,1)	(1,1,0)	(1,1,1)
(0,0,0)	(0,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(0,1,1)	(1,0,1)	(1,1,0)	(1,1,1)
(1,0,0)	(1,0,0)	(0,0,0)	(1,1,0)	(1,0,1)	(1,1,1)	(0,0,1)	(0,1,0)	(0,1,1)
(0,1,0)	(0,1,0)	(1,1,0)	(0,0,0)	(0,1,1)	(0,0,1)	(1,1,1)	(1,0,0)	(1,0,1)
(0,0,1)	(0,0,1)	(1,0,1)	(0,1,1)	(0,0,0)	(0,1,0)	(1,0,0)	(1,1,1)	(1,1,0)
(0,1,1)	(0,1,1)	(1,1,1)	(0,0,1)	(0,1,0)	(0,0,0)	(1,1,0)	(1,0,1)	(1,0,0)
(1,0,1)	(1,0,1)	(0,0,1)	(1,1,1)	(1,0,0)	(1,1,0)	(0,0,0)	(0,1,1)	(0,1,0)
(1,1,0)	(1,1,0)	(0,1,0)	(1,0,0)	(1,1,1)	(1,0,1)	(0,1,1)	(0,0,0)	(0,0,1)
(1,1,1)	(1,1,1)	(0,1,1)	(1,0,1)	(1,1,0)	(1,0,0)	(0,1,0)	(0,0,1)	(0,0,0)

Let the children verify all the five properties :

- The sum of any two triplets in the set is a triplet of the set.
- The associative law for addition holds.
- There exists an additive identity viz. (0,0,0) which when added to any triplet gives that triplet.
- For every element there exists an additive inverse. In fact every triplet is its own inverse (why?).
- The commutative law for addition holds.

Thus the eight triplets form a commutative additive group for the operation of addition defined here.

The children may now consider 16 quadruplets, define addition in the same way and verify that they also form an additive group.

These groups are very useful in information theory.

Experiment 268. Patterns in Multiplication of Columns.

The two fundamental operations in arithmetic are addition and multiplication. 'Subtraction' and 'division' are 'inverse' operations.

We examined some additive structures in the last six experiments. The children will normally like to explore the corresponding multiplicative structures and see whether similar properties hold here.

Let us consider first the multiplication of columns.

A	B	C	D	E
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30
..	..	..	..	..

How shall we define $A \times B$? Take *any* number from column A and *any* number from column B and multiply them. The product always lies in B, so that $A \times B = B$. Proceeding in this way, we get the multiplication table.

$\times$	A	B	C	D	E
A	A	B	C	D	E
B	B	D	A	C	E
C	C	A	D	B	E
D	D	C	B	A	E
E	E	E	E	E	E

Let the children examine the five properties given in Experiment 265 replacing 'addition' there by 'multiplication'.

- There is a set of five columns. The product of any two columns of this set is a column of the set.
- The associative law for multiplication holds e.g. $(A \times B) \times C = B \times C = A$; $A \times (B \times C) = A \times (A) = A$. The children can collectively verify the law for all the 125 cases.
- There exists a 'multiplicative identity' viz. column A. Any column multiplied by A gives that column. Also when A is multiplied by any column, we get the column.
- What about 'multiplicative' inverses? A multiplicative inverse of a column will be such a column that the product of the two columns is the multiplicative identity $A \times A = A$, $B \times C = A$, $C \times B = A$, $D \times D = A$. Therefore A and D are inverses of themselves and B and C are inverses of each other. However E does not have a multiplicative inverse, for there is no column which when multiplied by E gives A. The children may recall that E is the additive identity of additive group.
- The commutative property holds since $A \times B = B \times A$, $A \times C = C \times A$, $A \times D = D \times A$, $A \times E = E \times A$, $B \times C = C \times B$, $B \times D = D \times B$, $B \times E = E \times B$, $C \times D = D \times C$, $C \times E = E \times C$, $D \times E = E \times D$.

Thus all the properties of a commutative multiplicative group hold except that the additive identity E does not have a multiplicative inverse. The children may recall that even in ordinary numbers, 0 does not have a multiplicative inverse. If however E is left out, the remaining four elements form a commutative multiplicative group.

Let the children now consider six columns.

A	B	C	D	E	F
1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
...	...	...	...	...	...

The multiplication table is

$\times$	A	B	C	D	E	F
A	A	B	C	D	E	F
B	B	D	E	B	D	F
C	C	F	C	F	C	F
D	D	B	F	D	B	F
E	E	D	C	B	A	F
F	F	F	F	F	F	F

The first three properties still hold. A is the multiplicative identity, but elements B, C, D and F do not have multiplicative inverses. The commutative law holds. These six elements do not form a multiplicative group. Even if we leave out the additive identity F, the remaining elements do not form a multiplicative group.

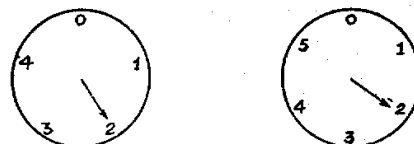
$$B \times D = F, C \times D = F.$$

F is the additive identity and corresponds to zero in our number system. In ordinary numbers if $b \times c = 0$ then either $b = 0$ or $c = 0$ or both are zero. Here neither B is the additive identity nor D is the additive identity, yet their product is the additive identity.

The children can easily verify that when the number of classes is a prime number i.e. when it is 3, 5, 7, 11, ..., the only element which does not have a multiplicative inverse is the additive identity and the product of two classes is the additive identity only if, at least one of them is the additive identity. If we leave out the additive identity, the remaining classes form a commutative multiplicative group. On the other hand if the number of classes is a composite number, there will be elements other than the additive identity which will not have inverses and there will be elements which are different from the additive identity whose product is the additive identity. Even if we leave out the identity element, the remaining elements will not form multiplicative groups.

Experiment 269. The Clock Arithmetic (Multiplicative).

For the clocks :



we have the multiplication tables :

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$\times$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

In both cases 0 is the additive identity and 1 is the multiplicative identity. In the first case the multiplicative inverses of 1, 2, 3, 4 are 1, 3, 2 and 4 respectively and only 0 does not have a multiplicative inverse. In this case non-zero elements form a commutative multiplicative group. In the second case 0, 2, 3, 4 do not have multiplicative inverses and

$$2 \times 3 = 0, 3 \times 4 = 0$$

$$\text{Also here } 3 + 4 = 1, 4 + 5 = 3$$

These equations do not show that the rules of our arithmetic are wrong. The quantities 0, 1, 2, 3, 4, 5 are not ordinary numbers since they have different rules for addition and multiplication.

Experiment 270. Multiplicative Group of Positive Fractions.

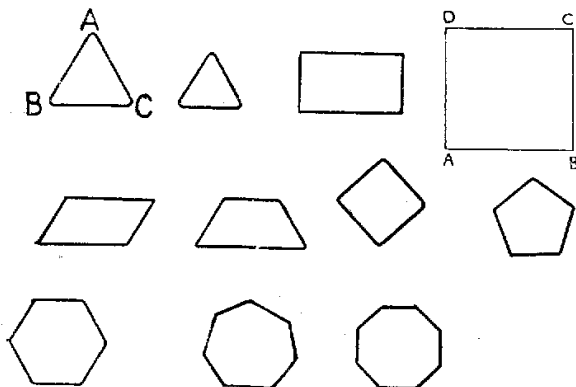
Consider the set of all positive fractions. We note that

- The product of two positive fractions is always a positive fraction.
- The associative law of multiplication holds. The children can verify it in some cases.
- There is a multiplicative identity viz. $\frac{1}{1}$ which when multiplied with any fraction gives that fraction.
- Every positive fraction has a multiplicative inverse e.g., the inverse of $\frac{2}{3}$ is $\frac{3}{2}$ and inverse of $\frac{5}{8}$ is $\frac{8}{5}$.
- The commutative law for multiplication holds.

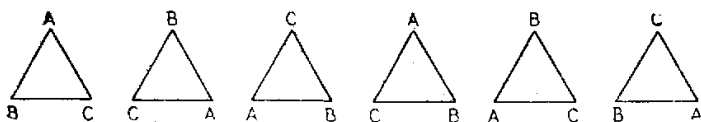
Thus positive fractions form a commutative multiplicative group. The children can easily show that natural numbers and whole numbers do not form multiplicative groups since no natural number or whole number except 1 has a multiplicative inverse and further the set of integers does not form a multiplicative group since no integers except 1 and -1 have multiplicative inverses.

Experiment 271. Groups Resulting from Hole Filling.

Take a piece of card-board and cut from it holes in the shapes of isosceles triangle, equilateral triangle, rectangle, square, parallelogram, trapezium, rhombus, regular pentagon, regular hexagon etc.



Let the children now try to fit in the pieces in the holes. In how many ways can each piece be fitted? Consider the equilateral triangle. It will fit the hole in each one of the following six ways:



We get the six transformations:

$$\begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}, \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}, \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}, \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix}, \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}, \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$

Consider the second and third transformations. The second transformation sends A into B and the third sends B again into A. Similarly the second transformation sends C into A and then

the third sends A into C. In this way we see that the second transformation followed by the third transformation gives the identity transformation.

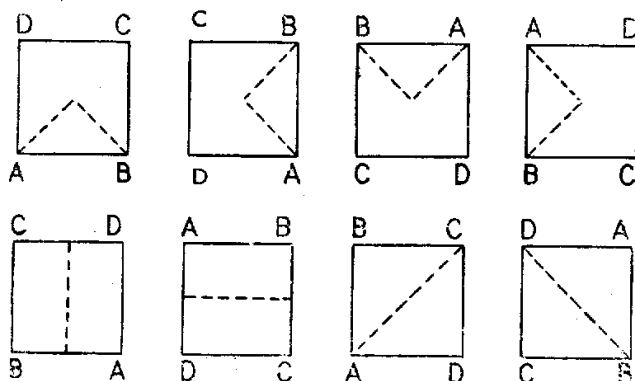
Further we have:

- (i) The product of two transformations is a transformation.
- (ii) The law of composition is associative

(iii) $\begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}$ is multiplicative identity.

(iv) Each transformation has a multiplicative inverse (verify). Therefore these six transformations form a multiplicative group.

Similarly let the children try to fit the square ABCD in its original hole. They will be able to fit it in 8 different ways:



The first four are obtained by rotating about a line through the centre perpendicular to the plane through $0^\circ, 90^\circ, 180^\circ, 270^\circ$ respectively and the last four are obtained by rotating about the dotted lines through 180° each. The names on the cardboard and the corresponding names in the piece replaced in the eight positions are given below:

$$\begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}, \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}, \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}, \begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix}$$

$$\begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}, \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}, \begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix}, \begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix}$$

These form a group.

Similarly the coincidence rotations of any regular polygon of n sides form a group with $2n$ elements.

The children in elementary classes may not appreciate the group idea but will certainly enjoy the hole-filling activity.

Experiment 272. Do Triplets form a Multiplicative Group?

Consider the triplets discussed in experiment 267 and let them be multiplied component-wise so that

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_1 b_1, a_2 b_2, a_3 b_3)$$

The multiplication table then comes out to be :

$\times$	(0,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(0,1,1)	(1,0,1)	(1,1,0)	(1,1,1)
(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,0)
(1,0,0)	(0,0,0)	(1,0,0)	(0,0,0)	(0,0,0)	(0,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
(0,1,0)	(0,0,0)	(0,0,0)	(0,1,0)	(0,0,0)	(0,1,0)	(0,0,0)	(0,1,0)	(0,1,0)
(0,0,1)	(0,0,0)	(0,0,0)	(0,0,0)	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,0)	(0,0,1)
(0,1,1)	(0,0,0)	(0,0,0)	(0,1,0)	(0,0,1)	(0,1,1)	(0,1,1)	(0,1,0)	(0,1,1)
(1,0,1)	(0,0,0)	(1,0,0)	(0,0,0)	(0,0,1)	(0,0,1)	(1,0,1)	(1,0,0)	(1,0,1)
(1,1,0)	(0,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(0,1,0)	(1,0,0)	(1,1,0)	(1,1,0)
(1,1,1)	(0,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(0,1,1)	(1,0,1)	(1,1,0)	(1,1,1)

The children can easily verify the following :

- The product of any two triplets is a triplet.
- The associative law for multiplication holds.
- (1, 1, 1) is the multiplicative identity.
- Except (1, 1, 1), no other element has an inverse.
- The commutative law for multiplication holds.

Due to (iv), the triplets do not form a group.

Experiments : 273 to 280

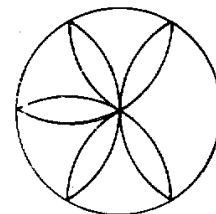
Experiment 273. Verifying Euler's Formula for Polyhedra.

In Experiment 87, the children have already verified Euler's formula $V - E + F = 2$ (when V is the number of vertices, E is the number of edges and F is the number of faces) in the following cases :

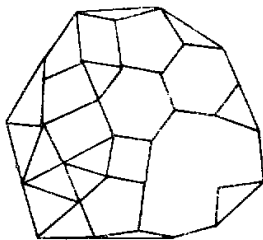
- Pyramids with bases as triangles, quadrilaterals, pentagons, hexagons etc.
- Prisms with bases as polygons of different number of sides.
- Combinations of pyramids and prisms.
- Other convex polyhedra.

The children may verify the formula again in all these cases. Each child may take a spherical balloon and draw a network on it with a pen and let him count the number of vertices, edges and faces and again verify Euler's formula. In this case, the edges and faces will, of course, be curved.

Each child may also draw a network on plain paper similar to those given on this and the next page consisting of closed curves triangles,

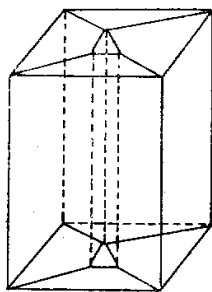
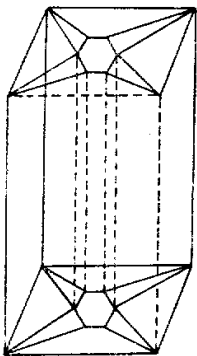


quadrilaterals, pentagons, etc., and let him count the number of vertices or corners, edges or lines and faces or regions. Let him



count the region outside the figure as one of the regions and again verify Euler's formula for this case. Let the children write all letters of the alphabet A, B, C,...Z, a, b, c,...z and find $V-E+F$ in each case.

Experiment 274. Euler's Formula for Polyhedra with One Hole.

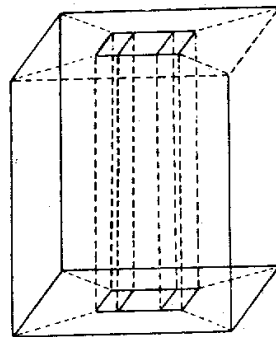


Let the children be given wooden cubical boxes with a triangular or quadrilateral or pentagonal hole in each case and let them count V, E, F e.g. they may get

	V	E	F	$V-E+F$
With a triangular hole	14	29	15	0
With a square hole	16	32	16	0
With a square hole	16	40	24	0
With a hexagonal hole	20	46	26	0

Even when the holes have the same number of sides, the number of edges and faces may be different, but $V-E+F$ would always come out to be zero.

Experiment 275. Euler's Formula for Polyhedra with Two or More Holes.



Let the children be given a cube with 2 holes (triangular, square, pentagonal or one triangular and the other square) and let them find V, E, F e.g. in the above figure

$$V=24, E=48, F=22,$$

so that

$$V-E+F=-2.$$

They will find in all cases whenever there are two holes, $V-E+F$ comes out to be -2 . Similarly for a polyhedron with three holes, $V-E+F$ comes out to be -4 . The children can now guess the formula

$$V-E+F=2-2q,$$

where q is the number of holes.

Experiment 276. Regular Solids.

A regular polyhedron is one in which there is no hole and for which the

(i) number of sides of each face is the same i.e. each face is a triangle or a quadrilateral or a pentagon etc.

(ii) number of edges which meet at a face is the same.

Let each face have m sides and let n edges meet at each vertex.

Each face has m edges so that F faces have mF edges, but in this, each edge is counted twice, so that

$$mF = 2E$$

Similarly

$$n \times V = 2E$$

But

$$V - E + F = 2$$

Therefore

$$\frac{2E}{n} - E + \frac{2E}{m} = 2$$

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{E} + \frac{1}{2}$$

We want m , n and E to be positive integers with m , n both ≥ 3 . By trial and error the children can find the following solutions only ;

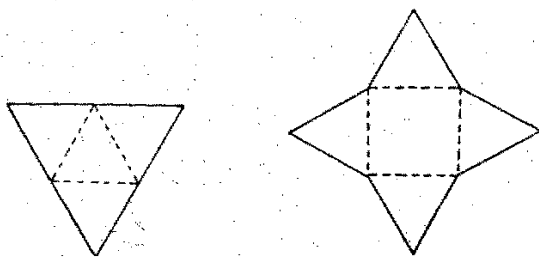
n	m	E	V	F	Polyhedron
3	3	6	4	4	Tetrahedron
4	3	12	6	8	Octahedron
5	3	30	12	20	Icosahedron
3	4	12	8	6	Cube
3	5	30	20	12	Dodecahedron

Let the children verify that $m=4, n=4$; $m=4, n=5$; $m=4, n=6$; $m=5, n=5$ etc. are not possible.

There are thus only five regular solids. In a regular solid we also usually require faces to be congruent figures.

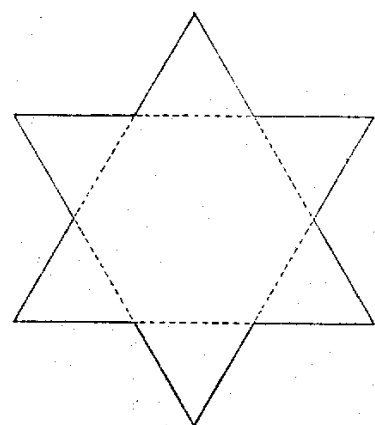
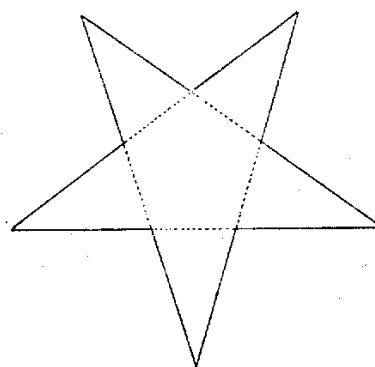
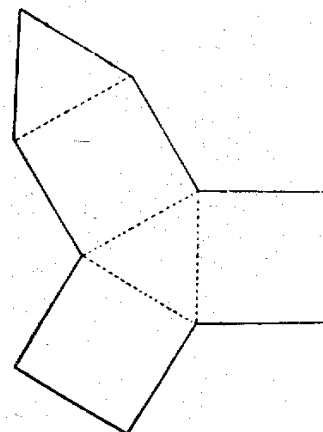
Experiment 277. Making Pyramids and Prisms.

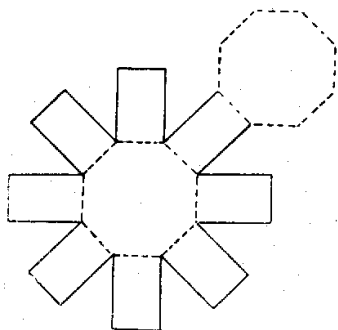
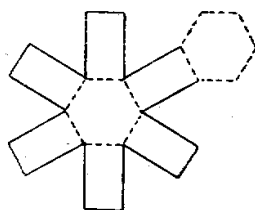
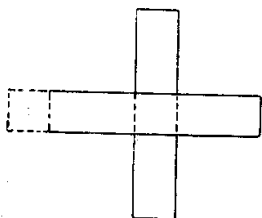
Let the children draw the following nets on cardboards and make pyramids by folding along dotted lines and then joining the edges.



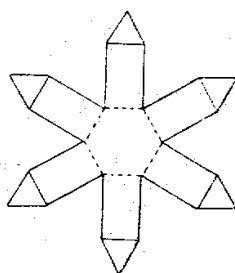
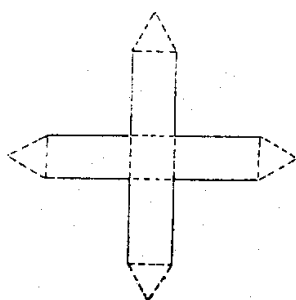
Similarly the children may make pyramids with a regular heptagon or a regular octagon as a base.

Next let them make prisms by drawing the following nets, folding along dotted lines and pasting together the edges.



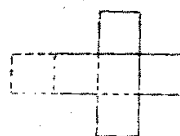


The children can draw also combinations of prisms and pyramids by drawing nets like the following :

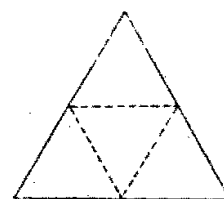


Experiment 278. Making Regular Polyhedra.

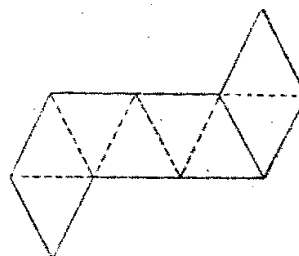
As we have seen there are five regular polyhedra. The nets for these are given below :



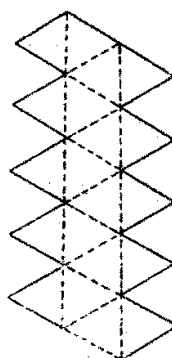
Cube



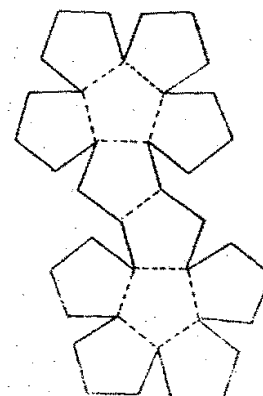
Tetrahedron



Octahedron



Icosahedron



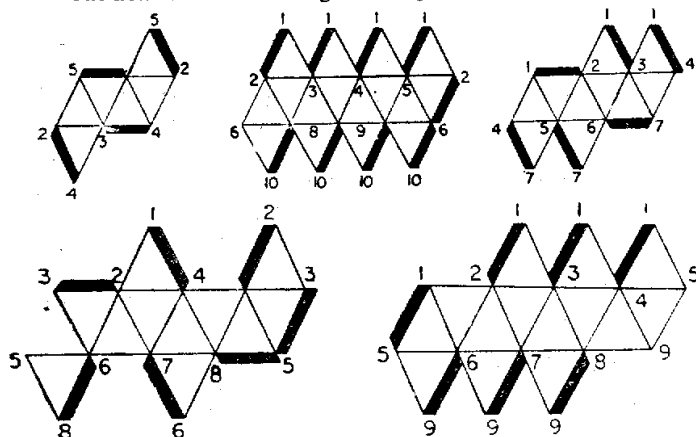
Dodecahedron

Experiment 279. Making Deltahedra.

These are regular convex polyhedra all of whose faces are congruent equilateral triangles. It was proved in 1947 that there are only eight such polyhedra and these are called deltahedra because each face is a triangle which is of the form of the Greek letter delta (Δ).

Three of these eight are the three regular polyhedra with triangular faces viz, tetrahedron, octahedron and icosahedron.

The nets for the remaining five are given below :



In these figures, after folding, the points with the same number have to be brought together. The shaded areas are to be used in pasting and joining.

Experiment 280. Motivation for learning geometry.

There can be two methods for making the above polyhedra. Either the nets can be drawn on good quality paper and children can copy these on card board by means of carbon paper and then they can do folding and pasting.

Alternatively the children can be motivated to learn the construction of equilateral triangles, squares, rectangles, regular pentagons, hexagons etc. in order to make these nets. Very often methods of constructing these are done in elementary classes and the possibility of constructing polyhedra with their help can be a strong motivating force for learning constructions in geometry.

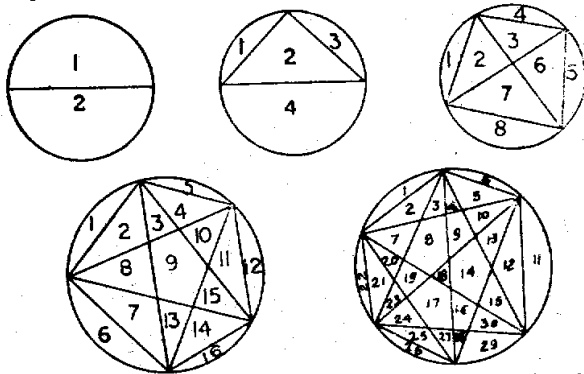
Experiments : 281 to 289

Experiment 281. Failure of Patterns.

(i) Give the children an expression like $n^2 - n + 41$ and let them give different values 1, 2, 3, 4, 5, 6,..... They get numbers 41, 43, 47, 53, 61, 71, 83, 113, 151,..... all of which happen to be prime numbers. The children may hazard a guess that all numbers would be prime numbers. Encourage them to continue and ultimately they would get a composite number. The bright children may be able to see even earlier that for $n = 41$, they would get a composite number. The children may learn the moral that while making intelligent guesses, through recognition of patterns is very useful; the guesses have to be confirmed through careful arguments. In the elementary stages, children should learn guessing, in later stages, they would learn how to 'prove' or 'disprove' the guesses.

(ii) Let the children draw a circle, take two points on it and join them. It divides the circular region into 2 parts. Let them now take three points and join every pair of these; the region is divided into 4 parts. Let them now take four points and find that the maximum number of regions they get is 8. With five points, they expect a maximum of 16 regions and they get them. With six points,

they expect to get a figure with 32 regions, but here the pattern fails.



The children are so much convinced of the pattern that they are prepared to believe they have missed some region and they try again and again. They believe the next number in the sequence 2, 4, 8, 16 must be 32. However consider

$$\frac{1}{24} (n^4 - 2n^3 + 11n^2 + 14n + 24)$$

When $n=1$, it gives 2; when $n=2$, it gives 4; when $n=3$, it gives 8; when $n=4$, it gives 16. However when $n=5$, it gives 31.

Given the first few terms of a sequence, we can find a large number of possibilities for subsequent terms.

The scientists guess at general patterns by examination of some particular cases. They go on confirming their guesses by further experimentation and they are sure about their guesses almost only when they get mathematical proofs.

The children are young scientists. Let them make shrewd guesses and go on verifying them. They have to learn deeper mathematics in order to get proofs of their guesses.

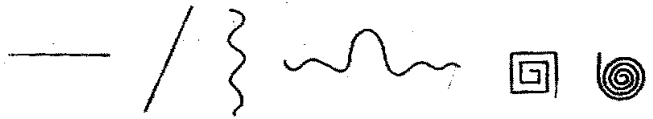
The children may also be asked to colour the regions obtained above with different colours in such a way that no two regions with common boundary get the same colour and let them verify that they don't require more than four colours in any case.

Experiment 282. Patterns in Estimation.

Children learn measures of lengths, areas, volumes and weights. A capacity to estimate these approximately is highly desirable. For

this the following experiments may be carried out.

(a) The teacher may draw some straight and curved lines on the board like the following :



and ask the children to guess the lengths in centimetres. Let the children then measure these lengths by means of a thread and find out how far they were wrong.

(b) Next let the children estimate the lengths and breadths of their class room, of the verandahs, the school hall, the playing fields etc and then by actual measurement find out their errors. In the same way let them measure the heights of children in the class and then find their errors. With practice, the estimates should go on improving.

(c) In the same way, children can by lifting different weights in their hands and then estimating their magnitudes in gms. and comparing them with the actual weights measured by, say, a spring balance, can develop the power of estimating weights correctly.

(d) The same method can be used to develop feeling for magnitude of areas in square cms. The children may draw many closed curves on graph paper, first estimate the number of small squares in each and then by actual counting, they can determine errors and thus reach better and better estimates in subsequent efforts.

(e) The children may similarly be asked to estimate how much time it takes to read a page, how much time it takes to run a furlong etc. and then by actual comparison, learn to improve their estimates. It will always be found that the first estimates are always wide off the mark, but they go on gradually improving.

(f) The children may be shown bottles of different sizes and asked to estimate their volumes in litres and then compare their estimates by filling the bottles with water and using a graduated cylinder.

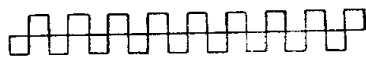
(g) The children may also estimate the number of children in the class, the number of children in the school, the number of children in their village or town, the population of the towns etc. and then compare these with the actual numbers.

Experiment 283. Further Patterns in Estimation.

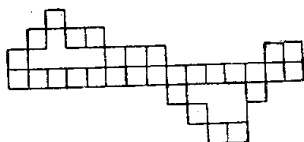
(a) Place a certain number of objects on the table and ask the children to estimate whether these are more than 100, more than 50, more than 25 etc.

(b) Draw patterns like the following on sheets of paper or on flannels and put them on the black board or flannel board and let the children estimate the number of squares and let them explain their method of getting these numbers.

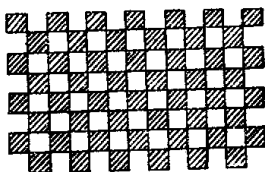
(i) The children can count only squares in the lower or upper line and then double the number.



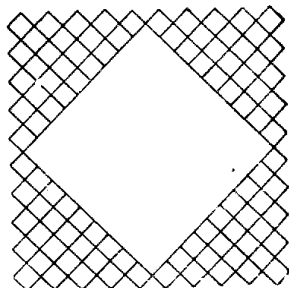
(ii) The children may see that there is one complete line of 15 squares and the other squares can be shifted to give another line of 15 squares.



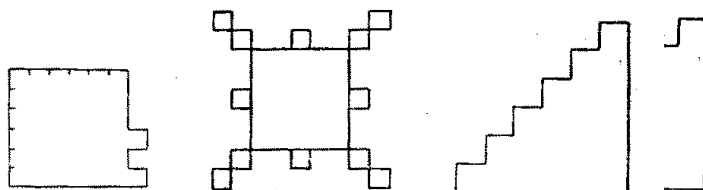
(iii) The children may see that the numbers in successive rows are 8, 7, 8, 7, 8, 7, 8, 7 and there are 4 pairs with 15 squares each.



(iv) The children may count the number of squares on one side and multiply by 4.



(c) The children may also estimate areas of figures like the following :



(d) The children may make stacks of centimetre cubes like the one shown here and estimate the total number of cubes in each stack.

10	5	4	3	3
7	4	2	1	1
8	6	4	2	2
5	3	2	1	1

(e) The children may estimate the square root of large numbers e.g., they may be asked whether the square root of 221456789321 will be more than 99999 or 111111 or less than 222121 etc. They may recall that the square of a number of n digits has either $2n-1$ or $2n$ digits. Similarly they may estimate and approximate cube roots or fourth roots.

(f) In the same way the children may estimate the products of quotients of numbers e.g., they may estimate how many digits the product 123456789×2345678 or quotient $\frac{123456789}{234}$ will have. This will be useful in checking errors in calculation.

(g) The children may be asked to estimate approximately the following. They may do these as oral exercises so that they are forced to use approximations :

(i) number of inches in a mile

$$(1760 \times 3 \times 12 \approx 1800 \times 35 = 63000)$$

(ii) number of square feet in an acre

$$4840 \times 9 \approx 4800 \times 9 = 43200$$

or

$$4840 \times 9 \approx 5000 \times 9 = 45000$$

- (iii) number of square inches in a circle of radius 10 feet.
- (iv) number of cubic inches in a sphere of diameter 10 feet.
- (v) number of Sundays in 200 years.

They may also be given books and asked to estimate the number of words and number of letters in them.

Experiment 284. The Square Bracket Function.

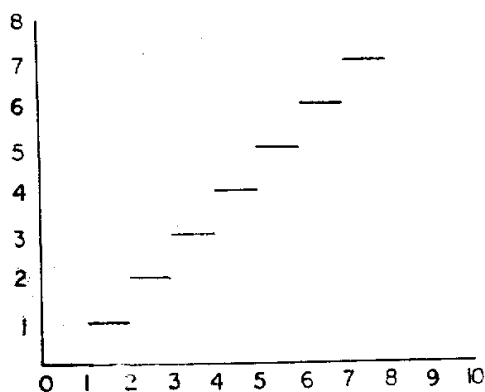
The square bracket function is defined as follows:

$[x]$ is the largest integer constrained in x .

Thus $[3\frac{1}{2}] = 3$, $[4\frac{5}{6}] = 4$, $[4.05] = 4$.

(a) The children may be given various numbers and asked to find their square brackets.

(b) The children may draw graph of this function e.g.



(c) The children may find $[x+y]$ and $[x] + [y]$ and investigate the relation between them. Thus

$$[3\frac{1}{4} + 6\frac{2}{3}] = [3\frac{1}{4}] + [6\frac{2}{3}]$$

$$[3\frac{1}{4} + 6\frac{2}{3}] > [3\frac{1}{4}] + [6\frac{2}{3}]$$

$$[6\frac{2}{3} - 3\frac{1}{4}] = [6\frac{2}{3}] - [3\frac{1}{4}]$$

$$[6\frac{2}{3} - 3\frac{1}{4}] < [6\frac{2}{3}] - [3\frac{1}{4}]$$

The children may also construct example of this type themselves and try to discover the general pattern.

(d) Similarly they may find relation between $[x \times y]$ and $[x] \times [y]$ and $[\frac{x}{y}]$ and $\frac{[x]}{[y]}$ e.g.

$$[\frac{6\frac{2}{3}}{3\frac{1}{4}}] = [\frac{6\frac{2}{3}}{3\frac{1}{4}}]$$

$$[\frac{6\frac{1}{4}}{3\frac{2}{3}}] < [\frac{6\frac{1}{4}}{3\frac{2}{3}}]$$

Children can make their own examples and try to deduce general results.

(e) The children may find relation below

$$[\sqrt{x}] \text{ and } \sqrt{[x]} \text{ e.g.}$$

$$[\sqrt{121.5}] = \sqrt{[121.5]}$$

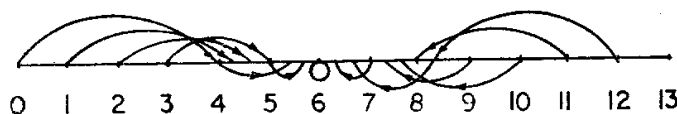
Will these be always equal?

Experiment 285. Patterns with Arrow Graphs.

(a) Let the children join points.

$$\square \text{ and } \frac{\square}{3} + 4$$

on the number line and see what they get.



They note that as the first point moves one step to the right, the second point moves one-third of a step to the right. Similarly when the first point moves one step to the left, the second point moves one-third of a step to the left. For points on the left of '6', the arrow points to the right, while for points on the right of 6, the arrow points to the left. The point '6' is joined to itself. We say this point remains fixed and it is the fixed point of this transformation.

(b) Let the children join points

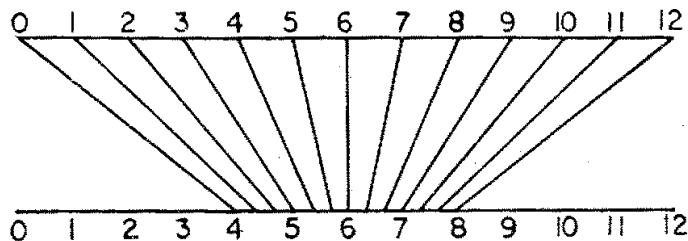
$$\square \text{ and } \frac{\square}{3} + 5$$

$$\text{or } \square \text{ with } 2\square - 3$$

$$\text{or } \square \text{ with } \frac{2\square}{5} + 1$$

and find the fixed points in all these and similar cases.

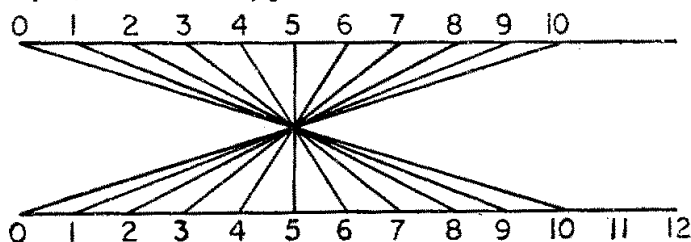
(c) Let the children draw two parallel lines and join points $\square$ on the first line with $\frac{\square}{3} + 4$ on the second line and let them find which



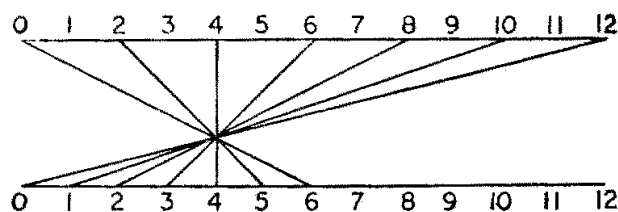
point on the first line goes into itself on the second line. This method gives a method of solving equations of the type

$$x = ax + b.$$

(d) Let the children join numbers whose sum is a given number, say 10, and see what they get :



They find all the lines passing through one point. The point of intersection shows the number which goes into itself. The above pattern corresponds to $\square + \triangle = 10$, and shows that the fixed point is 5. The following pattern shows $\square + 2\triangle = 12$ and shows that the fixed point is 4.



This gives a method of finding the intersection of

$$x + ay = b \text{ and } x = y.$$

(e) The children may write numbers on two parallel lines in reverse order and join $\square$ on one line to $2\square$ on the other or $\square$ on one line with $\frac{\square}{4}$ on the other and see what pattern emerges.

Experiment 286. Patterns with Magic Squares.

(a) One of the most famous magic square was given by Durer in one of his paintings.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Let the children note the rich variety of patterns here :

- (i) Sum of elements of each row is 34.
- (ii) Sum of elements of each column is 34.
- (iii) Sum of elements of each diagonal is 34.
- (iv) Sum of four corner elements is 34.
- (v) Sum of four central elements is 34.
- (vi) Sum of 4 elements in each 2×2 square into which the square is divided by central lines is 34.
- (vii) The opposite pairs of squares : 3, 2, 15, 14 ; 8, 12, 9, 5 add upto 34.
- (viii) The slanting pairs of squares : 2, 8, 9, 15 ; 3, 5, 12, 14 add upto 34.
- (ix) The sum of squares of elements of the first two rows = sum of squares of elements of the last two rows i.e.,
 $16^2 + 3^2 + 2^2 + 13^2 + 5^2 + 10^2 + 11^2 + 8^2$
 $= 9^2 + 6^2 + 7^2 + 12^2 + 4^2 + 15^2 + 14^2 + 1^2 = 748.$
- (x) The sum of squares of elements of first and third rows = sum of squares of elements of second and fourth rows i.e.,
 $16^2 + 3^2 + 2^2 + 13^2 + 9^2 + 6^2 + 7^2 + 12^2$
 $= 5^2 + 10^2 + 11^2 + 8^2 + 4^2 + 15^2 + 14^2 + 1^2 = 748$
- (xi) The sum of squares of elements of first and third columns = sum of squares of elements of second and fourth columns i.e.,
 $16^2 + 5^2 + 9^2 + 4^2 + 3^2 + 10^2 + 6^2 + 15^2$
 $= 2^2 + 11^2 + 7^2 + 14^2 + 13^2 + 8^2 + 12^2 + 1^2 = 748$

- (xii) The sum of squares of elements of first and third columns = sum of squares of elements of second and fourth columns i.e.,

$$16^2 + 5^2 + 9^2 + 4^2 + 2^2 + 1^2 + 7^2 + 14^2 \\ = 3^2 + 10^2 + 6^2 + 15^2 + 13^2 + 8^2 + 12^2 + 1^2 = 748$$

- (xiii) The numbers in the diagonals add upto the numbers not in the diagonals, and not only are the sum of their squares equal but the sum of their cubes are also equal. Thus

$$16 + 10 + 7 + 1 + 4 + 6 + 11 + 13 \\ = 2 + 8 + 12 + 14 + 15 + 9 + 5 + 3 = 68 \\ 16^2 + 10^2 + 7^2 + 1^2 + 4^2 + 6^2 + 11^2 + 13^2 \\ = 2^2 + 8^2 + 12^2 + 14^2 + 15^2 + 9^2 + 5^2 + 3^2 = 748 \\ 16^3 + 10^3 + 7^3 + 1^3 + 4^3 + 6^3 + 11^3 + 13^3 \\ = 2^3 + 8^3 + 12^3 + 14^3 + 15^3 + 9^3 + 5^3 + 3^3 = 9248$$

The verification of all these patterns will give the children well-motivated exercises in addition and multiplication.

- (b) The children may join the numbers in the magic square of Durer in increasing order and get the magic Durer line.

- (c) The following squares are also interesting

67	1	43		3	71	5	23
13	37	61		53	11	37	1
31	73	7		29	7	19	47
				17	13	41	31

Except for number 1, all other are prime. No composite number occurs in either square. In second square, only row and column sums are same.

- (d) The following magic square is also interesting and amazing

8818	1111	8188	1881
8181	1888	8811	1118
1811	8118	1181	8888
1188	8881	1818	8111

- (i) The sum of elements of every row, column, diagonal is 19998.

- (ii) If we write the square as

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

then

$$\begin{aligned} A + D + M + P &= 19998 \\ B + C + N + O &= 19998 \\ E + I + H + L &= 19998 \\ A + B + E + F &= 19998 \\ B + F + C + G &= 19998 \\ C + G + D + H &= 19998 \\ E + I + F + J &= 19998 \\ F + J + G + K &= 19998 \\ G + K + H + L &= 19998 \\ I + M + J + N &= 19998 \\ J + N + K + O &= 19998 \\ K + L + O + P &= 19998 \end{aligned}$$

- (iii) If we turn this square upside down, we still get a magic square with all the properties unchanged.

- (iv) If we hold this square in front of a mirror, we get another magic square in which all these properties remain unchanged.

- (v) If we turn the square upside down and hold it in front of a mirror, the same properties hold.

The children can get practice in more than 100 addition sums, in each of which the sum comes out to be 19998.

The magic square is called I X O X I O X I (pronounced Ixohocksie) because this name is the same backward or forward, upside down, in the mirror, in the mirror and upside down).

- (a) From a given magic square, the children can make new magic squares by interchanging rows or interchanging columns or adding the same number to every element or subtracting the same number from every element or multiplying every element by same number or dividing every element by same number or dividing by a given number and taking the remainders. By subtracting a suitable number from every element, the children can get magic squares with some positive and some negative elements. Similarly by dividing by a suitable number every element, they can get squares whose elements are fractions. The teacher can give such examples for giving

some practice in addition of fractions e.g., the following are magic squares :

$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{11}{12}$
$\frac{1}{3}$	$\frac{1}{4}$	$\frac{5}{6}$	$\frac{8}{12}$
$\frac{7}{6}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{11}{12}$
$\frac{7}{12}$	1	$\frac{1}{12}$	$\frac{1}{6}$

$\frac{11}{8}$	$\frac{1}{2}$	$\frac{5}{8}$
$\frac{3}{4}$	1	$\frac{3}{4}$
$\frac{7}{8}$	$\frac{3}{4}$	$\frac{5}{8}$

Experiment 287. Patterns in a Square.

(a) Give the children the following square

509	681	366	603	721
356	528	213	450	568
259	431	116	353	471
365	537	222	459	577
144	316	1	238	356

Let the children choose five numbers, one number being chosen from each row and each column. Thus let them first choose any one of the twenty-five numbers. Let them cross out the row and the column containing this number and let them choose one number out of the remaining sixteen numbers. Then let them cross out the row and the column containing this new number and let them choose one number from the remaining nine numbers and so on. Ultimately they will get 5 numbers. Let them add these five numbers. The sum comes out to be 1968 in whatever way the choice is made and there are $25 \times 16 \times 9 \times 4 \times 1 = 14400$ ways. The children can get practice in addition of five numbers in 14400 problems if they like. It would look surprising that the sum always comes out to be 1968.

(b) In the same way, the following square will give $36 \times 25 \times 16 \times 9 \times 4 \times 1 = 5184000$ addition sums in each of which the sum will come out to be 100

6	8	5	10	18	15
9	11	8	13	21	18
12	14	11	16	24	21
14	16	13	18	26	23
15	17	14	19	27	24
18	20	17	22	30	27

(c) The children will like to know how they can form such squares themselves for any given sum. Suppose they want 6×6

square with sum 100. They just choose 12 numbers whose sum is 100 e.g. they can choose :

2, 5, 8, 10, 11, 14
4, 6, 3, 8, 16, 13

and write these in two sets. Let them now add 2 to every element of the second set to get first row of square and add 5 to every element of this second set to get second row of square and so on.

2+4	2+6	2+3	2+8	2+16	2+13
5+4	5+6	5+3	5+8	5+16	5+13
8+4	8+6	8+3	8+8	8+16	8+13
10+4	10+6	10+3	10+8	10+16	10+13
11+4	11+6	11+3	11+8	11+16	11+13
14+4	14+6	14+3	14+8	14+16	14+13

They can now see why they always get the sum 100. In whatever way they choose the six numbers, they would get each of the twelve numbers 2, 5, 8, 10, 11, 14, 4, 6, 3, 8, 16, 13 once and once only and thus the sum must be the sum of these twelve numbers and must therefore be 100.

(d) What we have done for sums can be done for products also e.g. let the children form the squares :

2×4	2×6	2×3	2×8	2×16	2×13
5×4	5×6	5×3	5×8	5×16	5×13
8×4	8×6	8×3	8×8	8×16	8×13
10×4	10×6	10×3	10×8	10×16	10×13
11×4	11×6	11×3	11×8	11×16	11×13
14×4	14×6	14×3	14×8	14×16	14×13

i.e.

8	12	6	16	32	26
20	30	15	40	80	65
32	48	24	64	128	104
40	60	30	80	160	130
44	66	33	88	176	143
56	84	42	112	224	182

If the children choose six numbers, one from each row and each column and find the product, the product will be $2 \times 5 \times 8 \times 10 \times 11 \times 14 \times 4 \times 6 \times 3 \times 8 \times 16 \times 13 = 15260345600$ and they can get this product in 518400 ways.

The children can now form their own squares with given sum or product and given size and check the result.

Experiment 288. Patterns with Fractions.

(a) The children may make their 'fractions boards' like one as given below :

$\frac{1}{2}$		$\frac{1}{2}$	
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

This may be used to give the following concepts :

(i) Concept of equivalent fractions e.g. that

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \dots$$

$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} = \dots$$

(ii) Concept of Egyptian fractions whose numerators are restricted to 1 except in the case of the fraction $\frac{2}{3}$. The children may be asked to express any given fraction in terms of Egyptian fractions. Thus

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$$

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$$

$$\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$$

$$\frac{2}{5} = \frac{1}{4} + \frac{1}{10} + \frac{1}{20}$$

(iii) Concept of simple addition and subtraction of fractional parts.

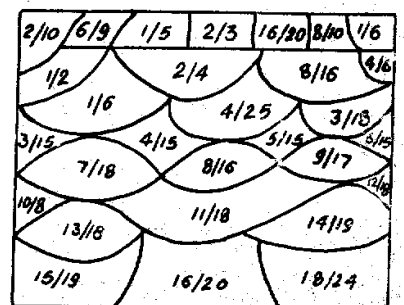
Thus

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

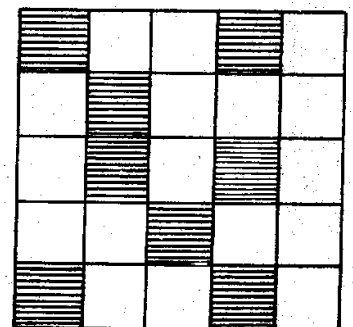
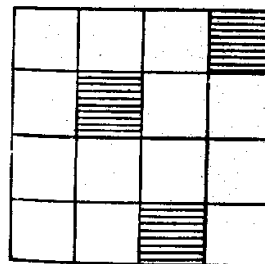
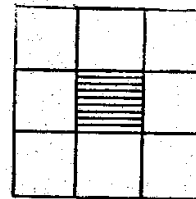
$$\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$$

(iv) Concept of the relative sizes of fractions.

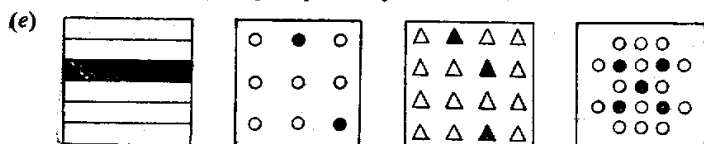
(b) In figures like the one shown here, the children may shade all those parts having fractions in their lowest terms or they may show equivalent fractions with the same colour.



(c) Children may write fractions indicating the shaded parts.



(d) Children may form magic squares of fractions by dividing elements of ordinary magic squares by same number.



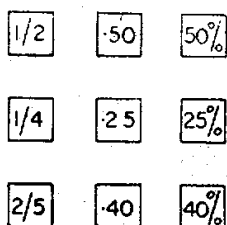
Children may be given cards like the above and they may be asked to arrange them in decreasing order of shaded fractions.

(f) Let the children find the number of positive proper fractions with denominators less than 10 or 20 or 30 and see the pattern. They may start with

- number of proper fractions with denominator two = 1
- number of proper fractions with denominator three = 2
- number of proper fractions with denominator four = 3.

They may also find the number of proper fractions in their lowest terms with denominator less than 10 or 20 or 30 etc.

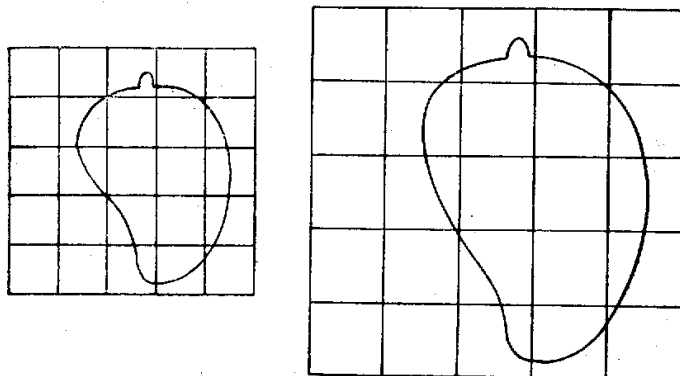
(g) Children familiar with decimals, fractions and percentages can play the game of Fradey Percey. There is a set of 54 cards but there may be more or less. Some cards are illustrated below.



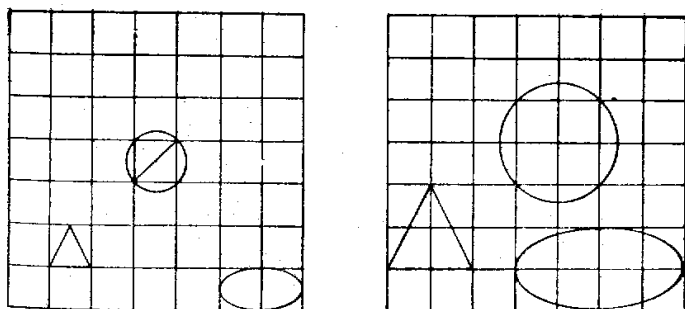
Cards are shuffled and distributed to 2 or 3 or 6 children. Each child draws a card from one on his left and as soon as he gets three matching cards, he lays them on the table and gets 10 points. If he lays down wrongly, the first child pointing the mistake gets 10 points. The game ends where one player is left with no cards and at that stage the score of each child is calculated as follows: 10 points for each triplet made are added and 2 points for each card remaining in hand are subtracted.

Experiment 289. Patterns in Enlargement.

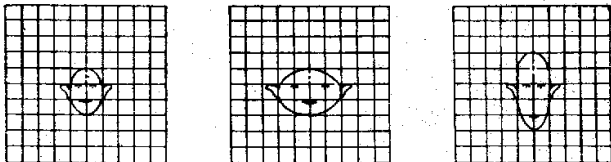
(a) Children like to enlarge given figures. To enlarge a given figure they have to draw a square grid on the original figure and then they reproduce the figure on a grid of different size.



The sheets of paper with square grids of different sizes may be printed for the children. On one sheet figures may be drawn and the children may enlarge these on different sheets to different sizes. Alternatively grids of the same size may be used but for enlarging, children may use two or three grids where they used one grid before.



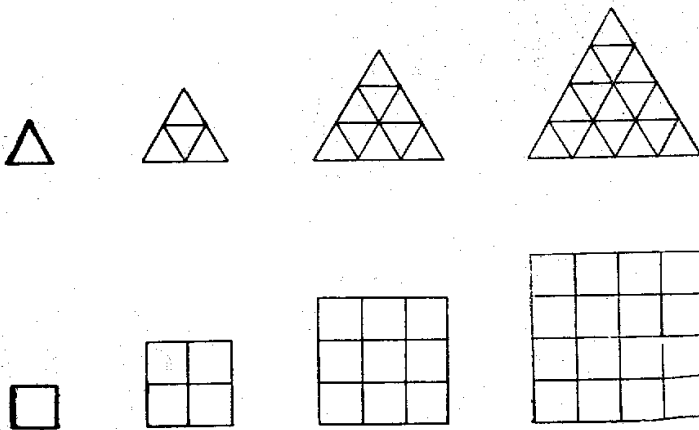
(b) Children can get interesting caricatures by using different scales of enlargement in different directions e.g. they may change the scale in one direction and not in the other or enlarge twice in one direction and three times in the other.



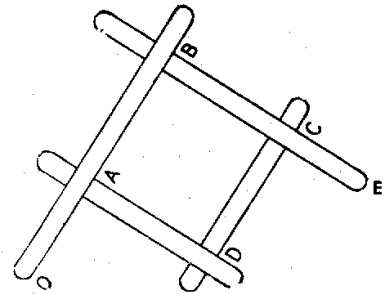
Children may like to caricature figures of important personalities of other shapes. They may also realise that for true representation, the scale of enlargement should be same in the two directions.

(c) Children can draw figures on surfaces of balloons and see how these get enlarged as the balloons are gradually filled with air or they may draw figures on flat pieces of rubber and see how these figures get deformed as the rubber piece is stretched in different directions.

(d) For some shapes, enlargements can be made by using 4 or 9 or 16 of them together e.g.

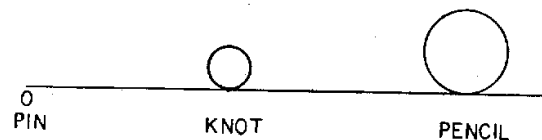


(e) The children can make a pantograph from their Meccano or strips of stout card



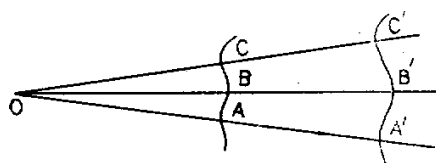
ABCD is a parallelogram and O, D, E are in a straight line. O is kept fixed. The point D moves over the figure to be enlarged and then pencil through E would trace out the enlarged figure. If the figure is to be shortened, E moves over the figure and D then traces out the smaller figure. Children like to make their own pantographs and make enlargements of figures chosen by them.

(f) A similar and simpler device is the 'elastic pantograph'.



Two elastic bands of the same length are knotted together. A drawing pin is inserted at one corner of one of the bands and acts as the enlarging centre. The knot is made to move over the figure to be enlarged and a pencil at the another end draws out a figure of double the dimensions. The bands are to be kept completely stretched in this process. If the size is to be tripled, we can use three bands.

(g) Another method of enlargement is the spider method.



From a point O, lines are drawn to points of the figure to be enlarged and the lines are extended to increase their lengths in the ratio of enlargement.

19

Miscellaneous Patterns II

Experiments : 290 to 295

Experiment 290. Patterns in Binary Scale Age Tables.

The children have learnt the binary scale in experiments 97 to 100. In this scale, every number is represented in terms of two digits 0 and 1.

Thus

$$(1101)_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = (13)_{10}$$

Conversely since

2	13
2	6 - ①
2	3 - ①
	① - ①

2	17
2	8 - ①
2	4 - ①
2	2 - ①
	① - ①

we write

$$(13)_{10} = (1101)_2,$$

$$(17)_{10} = (10001)_2$$

Thus every number in scale of 10 can be expressed in the binary scale. Let the children express all numbers upto 63 in the binary scale and write them on different cards according to the following rule :

Write all numbers which have in the binary scale '1' in the r th place from the right on the r th card.

Here $r=1, 2, 3, 4, 5, \dots$. Thus they get the following cards :

1	3	5	7	9	11
13	15	17	19	21	23
25	27	29	31	33	35
37	39	41	43	45	47
49	51	53	55	57	59
61	63				

2	3	6	7	10	11
14	15	18	19	22	23
26	27	30	31	34	35
38	39	42	43	46	47
50	51	54	55	58	59
62	63				

4	5	6	7	12	13
14	15	20	21	22	23
28	29	30	31	36	37
38	39	44	45	46	47
52	53	54	55	60	61
62	63				

8	9	10	11	12	13
14	15	24	25	26	27
28	29	30	31	40	41
42	43	44	45	46	47
56	57	58	59	60	61
62	63				

16	17	18	19	20	21
22	23	24	25	26	27
28	29	30	31	48	49
50	51	52	53	54	55
56	57	58	59	60	61
62	63				

32	33	34	35	36	37
38	39	40	41	42	43
44	45	46	47	48	49
50	51	52	53	54	55
56	57	58	59	60	61
62	63				

The children may note some patterns here e.g.

- (i) Each card has 32 numbers.
- (ii) The cards begin with numbers 1, 2, 4, 8, 16 and 32.
- (iii) The third card starts with 4, has 4 consecutive integers viz 4, 5, 6, 7, then there is a jump of 5 ($=4+1$) and then there are 4 consecutive integers again and then there is a jump of 5 and then there are 4 consecutive integers and so on. Similarly the fourth card begins with 8, has eight consecutive integers, then there is a

gap of 9 and then there are eight consecutive integers and so on. The children see the same pattern in every card. There are as many consecutive integers as the first number followed by a jump of one more than this number. Noting these patterns the children can prepare 7 cards for numbers upto 127 or 8 cards for numbers upto 255 and so on.

- (iv) The children may also notice that the first card contains all numbers of the form $2n-1$, second card contains all numbers of the form $4n-2, 4n-1$, third card contains all numbers of the form $8n-4, 8n-3, 8n-2, 8n-1$, fourth card contains all numbers of the form $16n-8, 16n-7, 16n-6, 16n-5, 16n-4, 16n-3, 16n-2, 16n-1$ and so on.

These cards can be used in a number of ways e.g.

- (i) These can be used for telling the ages of persons. A child asks a person to tell him which cards carry a number equal to his age, adds the first numbers on the cards and tell him immediately his age. In fact the child needs only see the backs of the cards, which may carry some distinct marks like different colours to see which numbers they start with.
- (ii) These can be used for 'think of a number' game. A person can think of a number and indicate on which cards the number occurs. The child easily tells which number he had thought of.
- (iii) These can be used for signalling messages e.g. the first, second, third, fourth, fifth, sixth and seventh cards may be represented by violet, indigo, blue, green, yellow, orange and red coloured bulbs (these are the colours in the spectrum of white light and are represented by the word (VIBGYOR) and suppose we have to transmit a word PATTERN. It has sixteenth, first, twentieth, twenty fifth, eighteenth and fourteenth letters of alphabet. We have to flash 16, 1, 20, 20, 5, 18, 14. Since 16 occurs on fifth card only, we flash yellow light and then flash a white light to show that the first letter is over. 1 occurs on the first card only so we flash violet light and then white light. 20 occurs on third and fifth cards, so we flash blue, yellow and white lights and so on. The message will then become

YVWBYWBYWVBWYIWIBGWW

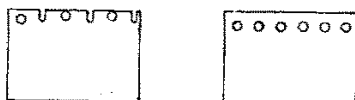
For flashing messages in English, we shall require bulbs of five different colours and a white bulb. A white bulb light indicates

end of a letter and two white lights will indicate the end of a word.

- (iv) If cards are available, numbers can be expressed in binary scale. Thus 21 occurs in first, third and fifth cards. In the binary scale, digit 1 will occur in the first, third and fifth places from the right and 0 will occur in second and fourth places, so that

$$(21)_{10} = (10101)_2$$

- (v) These cards can be used for weighing. A person may have available weights of 1, 2, 4, 8, 16 and 32 Kgms. only. Suppose he wants to weigh 41 kgms. He finds 41 occurs on the first, fourth and sixth cards. He uses the first, fourth and sixth weights
- (vi) A similar use is the following:
A farmer may keep 1, 2, 4, 8, 16, 32 sheep in different pens. Suppose a buyer wants 41 sheep, he just empties first, third and sixth pens.
- (vii) These can be used for preparing a mechanical sorter of cards. Suppose we have 64 cards. We punch 6 holes on each as follows and number these cards and make cuts on a card according to its number. The card with number 41 is cut as the card on the left:



Now suppose the cards have got mixed up and we want to re-arrange them. We put a nail through the first hole on the right and shake. 32 cards will remain on the nail. Place these on the nail in front of others. Now put the nail in the second hole and shake and put those that remain in front on the nail. After six shakings, the cards would have been arranged in the natural order.

Experiment 291. Another Pattern in Binary Scale.

- (i) Here the children can learn the 'Russian peasant' method of multiplication which is based on the binary scale. To multiply

84 by 35, we proceed as follows :

	84	35	
	42	70	
*	21	140	140
	10	280	
*	5	560	560
	2	1120	
*	1	2240	2240
			2740

The children go on dividing numbers in the first column by 2 successively disregarding remainders and marking the odd quotients by stars. The children go on multiplying numbers in the second column by 2 successively and finally add those numbers in the second column which correspond to odd numbers in the first column. The sum will give the required product.

The children can verify the result in a number of cases.

To understand the reason, let us express 84 in the binary scale.

$$84 = 64 + 16 + 4 = 2^6 + 2^4 + 2^2 \text{ so that}$$

$$(84)_{10} = (1010100)_2$$

$$\begin{aligned} 84 \times 35 &= \{1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0\} \times 35 \\ &= 2^6 \times 35 + 2^4 \times 35 + 2^2 \times 35 \\ &= 2240 + 560 + 140 = 2940. \end{aligned}$$

Thirty-five is to be multiplied by 2, twice in succession, four times in succession and six times in succession and these products are to be added. This is what has been done in the above example.

The advantage of this method is that the child can multiply any two numbers by using the multiplication table of 2 only. The disadvantage is that it takes more time.

- (ii) A modification of this method is obtained by expressing both numbers in binary scale.

$$(84)_{10} \times (35)_{10} = (1010100)_2 \times (100011)_2$$

$$(84)_{10} \times (35)_{10} = (2^6 + 2^4 + 2^2) \times (2^5 + 2 + 1)$$

$$= 2^{11} + 2^9 + 2^7 + 2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2$$

$$= 2^{11} + 2^9 + 2^8 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2$$

$$= (2940)_{10}$$

Alternatively we note the following :

Number 84 occurs on 7th, 5th, and 3rd cards

Number 35 occurs on 6th, 2nd and 1st cards.

To find the product, we add first numbers in 12th, 8th, 7th, 10th, 6th, 5th, 8th, 4th and 3rd cards.

The rule that emerges is the following :

Let first number occur in a th, b th, c th...cards, and let the second number occur in p th, q th, r th...cards ; then the product is the sum of first numbers on the following cards :

$$(a+p-1)\text{th}, (a+q-1)\text{th}, (a+r-1)\text{th}, (b+p-1)\text{th}, \\ (b+q-1)\text{th}, (b+r-1)\text{th}, \dots$$

This method has the advantage that one has not to know even the multiplication table of 2. What one requires is the addition of numbers only. Of course we have to have sufficient cards and sufficient time. What we gain in simplicity, we lose in time.

(iii) Another notation may be useful. Since

$$84 = 2^6 + 2^4 + 2^1 \\ 35 = 2^5 + 2^1 + 2^0,$$

we write

$$84 = (6, 4, 2), \quad 35 = (5, 1, 0)$$

The children may express various numbers in this form.

The method for multiplication is illustrated below :

$$84 \times 35 = (6, 4, 2) \times (5, 1, 0) \\ = (6+5, 6+1, 6+0, 4+5, 4+1, 4+0, 2+5, 2+1, 2+0) \\ = (11, 7, 6, 9, 5, 4, 7, 3, 2) \\ = (11, 9, 8, 6, 5, 4, 3, 2) \\ = 2^{11} + 2^9 + 2^8 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 = 2940.$$

In the above simplification, whenever a number like 7 occurs twice, we replace the pair by the next number. Thus

$$(11, 10, 10, 9, 9, 9, 7, 7, 7, 7) = (11, 11, 10, 9, 8, 8) \\ = (12, 10, 9, 9) \\ = (12, 10, 10) \\ = (12, 11).$$

This notation may also be useful for addition and subtraction. These are illustrated below :

$$84 + 35 = (6, 4, 2) + (5, 1, 0) = (6, 5, 4, 2, 1, 0) \\ = 2^6 + 2^5 + 2^4 + 2^2 + 2^1 + 2^0 \\ = 119.$$

$$84 - 35 = (6, 4, 2) - (5, 1, 0) = (5, 5, 4, 1, 1) - (5, 1, 0) \\ = (5, 4, 1) - (0) = (5, 4, 0, 0) - (0) = (5, 4, 0) \\ = 2^5 + 2^4 + 2^0 = 32 + 16 + 1 = 49.$$

The notation may be used for direct multiplication, addition and subtraction in the binary scale. Thus

$$(1101100101)_2 \times (1001001010)_2 \\ = (9, 8, 6, 5, 2, 0) \times (9, 6, 3, 1) \\ = (18, 15, 12, 9, 17, 14, 11, 9, 15, 12, 9, 7, 14, 11, 8, 5, \\ 11, 8, 5, 3, 9, 6, 3, 1) \\ = (18, 17, 16, 15, 13, 12, 11, 9, 8, 5, 3, 2, 1) \\ = (1111011101100101110)_2 \\ (1101100101)_2 + (1001001010)_2 \\ = (9, 8, 6, 5, 2, 0) + (9, 6, 3, 1) \\ = (9, 9, 8, 6, 6, 5, 3, 2, 1, 0) \\ = (10, 8, 7, 5, 3, 2, 1, 0) \\ = (10110101111)_2 \\ (1101100101)_2 - (1001001010)_2 \\ = (9, 8, 6, 5, 2, 0) - (9, 6, 3, 1) \\ = (8, 4, 3, 3, 1, 1, 0) - (3, 1) \\ = (8, 4, 3, 1, 0) = (100011011)_2$$

Experiment 292. Patterns in Ternary Scale.

(i) In Russian peasant method, the children have to use the multiplication table of 2 only, but it takes a long time. If they are prepared to use the multiplication tables of both 2 and 3, they may proceed as follows :

**86	35	70
*28	105	105
9	315	
3	945	
1	2835	2835
		3010

In the first column, the children divide successively by 3 and write two stars against a number which gives 2 as remainder and one

star against a number which gives 1 as remainder on division by 3. Let them multiply the numbers in the second column by 1 or 2 according as there are one or two stars in the corresponding numbers in the first column and then add.

Let them verify this algorithm in a number of cases.

The reasoning is explained by using the ternary scale

$$86 = 3^4 + 3^1 + 2 \times 3^0 = (1\ 00\ 1\ 2)_3$$

$$86 \times 35 = 3^4 \times 35 + 3^1 \times 35 + 2 \times 3^0 \times 35$$

$$= 2835 + 105 + 70 = 3010$$

$$35 = 3^3 + 2 \times 3^1 + 2 \times 3^0$$

The children may also use the following notation :

$$86 = (4, 1, 0, 0), \quad 35 = (3, 1, 1, 0, 0)$$

They multiply and add as before ; however when they get three equal numbers, they replace it by the next number. Thus

$$86 \times 35 = (4, 1, 0, 0) \times (3, 1, 1, 0, 0)$$

$$= (7, 5, 5, 4, 4, 2, 2, 1, 1, 3, 1, 1, 0, 0, 3, 1, 1, 0, 0)$$

$$= (7, 6, 4, 2, 1, 0)$$

$$= 3^7 + 3^6 + 3^4 + 3^2 + 3^1 + 3^0 = 3010$$

$$86 + 35 = (4, 1, 0, 0) + (3, 1, 1, 0, 0)$$

$$= (4, 3, 1, 1, 1, 0, 0, 0, 0)$$

$$= (4, 3, 2, 1, 0) = 3^4 + 3^3 + 3^2 + 3^1 + 3^0$$

$$= 121.$$

This notation also enables the children to multiply, add and subtract in ternary notation directly e.g.

$$(2110202)_3 \times (1220102)_3$$

$$= (6)_2, 5, 4, (2)_2, (0)_2 \times 6, (5)_2, (4)_2, 2, (0)$$

$$= (12)_2, (11)_4, (10)_4, (8)_2, (6)_4, (11)_2, (10)_2, (9)_2, 7, (5)_2, (0)$$

$$(9)_2, (8)_2, 6, (4)_2, (8)_2, (7)_4, (6)_4, (4)_2, (2)_4, (6)_2, (5)_4, (4)_4, (2)_2, (0)_4$$

$$= (13)_1, (12)_1, (11)_2, (9)_1, (6)_2, (5)_1, (3)_1, (2)_1, (0)_1$$

$$= (11201001201111)_3$$

$$(2110202)_3 + (1220102)_3$$

$$= (6)_2, 5, (4), (2)_2, (0)_2$$

$$+ (6)_1, (5)_2, (4)_2, 2, (0)_2$$

$$= (6)_3, (5)_3, (4)_3, (2)_3, (0)_4$$

$$= (7, 6, 5, 3, 1, 0)$$

$$= (11\ 1\ 0\ 1\ 0\ 11)_3$$

(ii) One important puzzle of the binary scale is the Baskets weight problem. The children may express every number upto 40, say 4, in ternary scale. Thus we have

Decimal	1	2	3	4	5	6	7	8	9	10
Ternary	1	2	10	11	12	20	21	22	100	101
Decimal	11	12	13	14	15	16	17	18	19	20
Ternary	102	110	121	112	120	121	122	200	201	202
Decimal	21	22	23	24	25	26	27	28	29	30
Ternary	210	211	212	220	221	222	1000	1001	1002	1010
Decimal	31	32	33	34	35	36	37	38	39	40
Ternary	1011	1012	1020	1021	1022	1100	1101	1102	1110	1111

To weight any weight upto 40 Kgms. we need only four weights viz 1 kgm., 3 kgms., 9 kgms., 27 kgms.

$$\text{Since } 31 = (1011)_3, 32 = (1012)_3$$

to weight 31 kgms. we use weights of 27, 3 and 1 kgms., while to weight 32 kgms. we use weights of 27 and 9 kgms. on one side and of 1 and 3 kgms. on the other side.

The children may express all numbers upto 40 in terms of numbers 1, 3, 9, 27 and + and - signs. Thus

$$32 = 27 + 9 - 3 - 1 = (1\ 1\ -1\ -1)_3$$

$$35 = 27 + 9 - 1 = (1\ 10\ -1)_3$$

$$38 = 27 + 9 + 3 - 1 = (1\ 11\ -1)_3$$

Similarly they may also express all numbers upto 121 in terms of numbers 1, 3, 9, 27 and 81 and + and - signs. Here use of modified ternary scale is useful. The number

$$(a\ b\ c\ d)_3$$

in this scale means, $a \times 3^4 + b \times 3^3 + c \times 3^2 + d \times 3 + e \times 3^0$

where a, b, c, d, e may be any one of the three 1, 0 and -1.

Similarly the children may try to express all numbers upto 170 in the form

$$a \times 4^3 + b \times 4^2 + c \times 4^1 + d \times 4^0$$

where a, b, c, d may be -1, 0, 1, 2.

Experiment 293. Patterns in Factorization on Primes.

(a) Ask the children to find prime factors of some given numbers e.g. different children may find

$$180 = 2 \times 2 \times 3 \times 3 \times 5, \quad 180 = 2 \times 3 \times 5 \times 2 \times 3$$

$$180 = 5 \times 3 \times 2 \times 2 \times 3, \quad 180 = 3 \times 5 \times 3 \times 2 \times 2$$

Different children will get different factorizations but they will get the same factors, possibly in different orders. They may try with a large number of numbers and they will find that this pattern of unique factorization, except possibly for order, always persists.

Children may use the tests of divisibility given in Experiment 96 and first divide by 2 as many times as possible, then divide by 3 as many times as possible and in the same way use 5, 7, 11, 13, as divisors successively. They may also use the index notations e.g. they write

$$180 = 2^2 \times 3^2 \times 5^1, \quad 420 = 2^2 \times 3^1 \times 5^1 \times 7^1$$

$$315 = 3^2 \times 5^1 \times 7^1, \quad 900 = 2^2 \times 3^2 \times 5^2.$$

(b) Let the children multiply two numbers and express the product also in the index notation e.g.

$$180 \times 315 = 56700 = 2^2 \times 3^4 \times 5^2 \times 7^1$$

$$180 \times 420 = 75600 = 2^4 \times 3^3 \times 5^2 \times 7^1$$

$$315 \times 900 = 283500 = 2^2 \times 3^4 \times 5^3 \times 7^1$$

What pattern do they find? Let them discover that when they multiply two numbers, the indices are just added. They may be able to explain this themselves.

(c) Let the children find the H.C.F. of two or three numbers e.g.

Numbers	Numbers in index notation	H.C.F.	H.C.F. in index notation
180, 315	$2^2 \times 3^2 \times 5^1, 3^2 \times 5^1 \times 7^1$	45	$3^2 \times 5^1$
180, 420	$2^2 \times 3^2 \times 5^1, 2^2 \times 3^1 \times 5^1 \times 7^1$	60	$2^2 \times 3^1 \times 5^1$
180, 315, 420	$2^2 \times 3^2 \times 5^1, 3^2 \times 5^1 \times 7^1, 2^2 \times 3^1 \times 5^1 \times 7^1$	15	$3^1 \times 5^1$

What pattern do they find? In the first case 3^2 and 5^1 occur in both numbers and they also occur in H.C.F. In the second case 2^2 and 5^1 occur in both numbers and they also occur in the H.C.F. Moreover the first number has 3^2 and second number has 3^1 as factors and H.C.F. has 3^1 as a factor. Thus the H.C.F. has always the smaller index. Let them try to generalise the result.

(d) Let the children find the L.C.M. of two or three numbers e.g.

Numbers	Numbers in index notation	L.C.M.	L.C.M. in index notation
180, 315	$2^2 \times 3^2 \times 5^1, 3^2 \times 5^1 \times 7^1$	1260	$2^2 \times 3^2 \times 5^1 \times 7^1$
180, 420	$2^2 \times 3^2 \times 5^1, 2^2 \times 3^1 \times 5^1 \times 7^1$	1260	$2^2 \times 3^2 \times 5^1 \times 7^1$
180, 315, 900	$2^2 \times 3^2 \times 5^1, 3^2 \times 5^1 \times 7^1, 2^2 \times 3^2 \times 5^2$	6300	$2^2 \times 3^2 \times 5^2 \times 7^1$

In this case the factors of L.C.M. are obtained by taking the largest of the indices in the numbers. Let the children verify this in a large number of cases.

Experiment 294. Patterns in Interesting Numbers.

(a) **Ramanujan's number.** The famous mathematician Hardy who helped Ramanujan very much, once went to see Ramanujan who was ill. He said he had travelled in a taxi with number 1729 which appeared to be a dull number with no interesting properties. Ramanujan who was a friend of almost every number said, "No, it is very interesting number; it is the smallest number which can be expressed as the sum of two cubes in two different ways."

$$1729 = 10^3 + 9^3 = 11^3 + 12^3 = 1 + 1728$$

Let the children find these pairs and show that no smaller number has this property. Let the children also find the smallest number which can be expressed as the sum of two cubes in three different ways, as the sum of three cubes in two different ways and in general as the sum of m n th powers in p different ways, for small values of m, n, p . This will give them sufficient well-motivated exercise in multiplication and addition.

(b) **Perfect numbers.** The proper divisors of 6 are 1, 2, 3 and the sum of these divisors is 6. Similarly the proper divisors of 28 are 1, 2, 4, 7, 14 and their sum is also 28. A number which is equal to the sum of its proper divisors is called a perfect number. Ancients attached great importance to these numbers. Thus they explained the creation of the world in 6 days and rotation of the moon about the earth in 28 days because 6 and 28 are perfect numbers. Let the children find the next perfect number. It will give them sufficient practice in factorization, since the next number is 496 and one after that is 8128. The next one is 33,550,336. Let the children verify that these are perfect numbers. Nobody knows whether there is an odd perfect number or not.

(c) **Amicable numbers.** These are number pairs such that each number is equal to the sum of proper divisors of the other. Thus the number 220 has proper divisors 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110 and their sum is 284, while 284 has proper divisors 1, 2, 4, 71, 142 and their sum is 220. Thus 220, 284 give an amicable number pair. Let the children verify that 17296, 18416 and 9363584 and 9437056 are also amicable number pairs.

(d) **Twin primes.** Number pairs like (5, 7), (11, 13), (17, 19) are called twin primes because both numbers are prime and the difference is just 2. Let the children find 100 such pairs. Nobody has been able to prove that there is an infinity of such twin primes or not.

(e) **Mersenne and Fermat numbers.** These are numbers of the form $2^n - 1$ and $2^n + 1$ respectively. Let the children express these in binary scale and see the patterns that emerge. They get

Mersenne numbers

1
1 1
1 1 1
1 1 1 1
1 1 1 1 1

Fermat numbers

1 1
1 0 1
1 0 0 0 1
1 0 0 0 0 0 0 0 1
1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1

Can they write the 10th Fermat number in binary notation without first writing it in decimal form?

(f) **Personality of numbers.** Every number has a personality of its own. Some are prime, some are triangular, some are square, some are cube, some are fourth powers, some are perfect, some are products of primes with only first powers of every prime factor, some are sum of squares in 2 ways, some are sum of squares in 3 ways only. Let the children try to find some interesting property of every number from 1 to 100. In this way it appears that every number is interesting, because suppose all numbers less than n are interesting, then n has to be interesting because n is the first number which is not interesting.

Experiment 295. Patterns in Checking Calculations.

(a) **Casting out nines.** There is a story that a primitive chief used pebbles to count his sheep. He used to put a pebble for each sheep in the unit's place and when he reached ten, he cast nine pebbles and placed one pebble in the ten's place. Let the children use this method to count 154 matchsticks.

Next let the children learn how to find the remainder when a given number is divided by 9. Suppose the number is 537. They can write it as

$$\begin{aligned} 537 &= 7 + 3 \times 10 + 5 \times 100 = 7 + 3(9 + 1) + 5(99 + 1) \\ &= (7 + 3 + 5) + \text{a multiple of 9} \\ &= 15 + \text{a multiple of 9} \\ &= 5 + 1 \times (9 + 1) + \text{a multiple of 9} \\ &= 6 + \text{a multiple of 9} \end{aligned}$$

Thus the remainder is 6. They find that 15 is sum of digits of 573 and 6 is sum of digits of 15. They can do a number of more such examples, they add the digits of the numbers, they again add the digits of this sum till they get a number between 1 and 9. If the sum is 9, then the remainder is zero, otherwise it is the final number obtained.

Next let the children note that

(a multiple of 9 + one number) + (a multiple of 9 + another number) = a multiple of 9 + sum of these 2 numbers and that the same rule holds for other operations.

Thus to check the result of any operation, we can carry out the same operations on remainder and this will check the result. Thus

$$\begin{array}{rcl} 1234567 & \rightarrow & 28 \rightarrow 10 \rightarrow 1 \\ + 3215789 & \rightarrow & 35 \rightarrow 8 \rightarrow 8 \\ \hline 4450356 & \rightarrow & 27 \rightarrow 9 \rightarrow 9 \\ 1234567 & \rightarrow & 28 \rightarrow 10 \rightarrow 1 \\ \times 12 & \rightarrow & 3 \rightarrow 3 \rightarrow 3 \\ \hline 14814804 & \rightarrow & 30 \rightarrow 3 \rightarrow 3 \\ 3215789 & \rightarrow & 35 \rightarrow 8 \rightarrow 8 \\ - 1234567 & \rightarrow & 28 \rightarrow 10 \rightarrow 1 \\ \hline 1981222 & \rightarrow & 25 \rightarrow 7 \rightarrow 7 \\ 14441 & \rightarrow & 14 \\ \div 11 & \rightarrow & 2 \\ \hline 1231 & \rightarrow & 7 \end{array}$$

(b) **Casting out elevens.** Children often make the mistake of interchanging two consecutive digits. The method of casting out nines cannot find it out. Casting out elevens can find such errors.

To check using this method, the algebraic sum of digits is found when alternative signs are assigned with unit's digit always positive.

$$\begin{array}{rcl} 34567 & \rightarrow & 7 - 6 + 5 - 4 + 3 \rightarrow 5 \rightarrow 5 \\ + 23456 & \rightarrow & 6 - 5 + 4 - 3 + 2 \rightarrow +4 \rightarrow +4 \\ \hline 58023 & \rightarrow & 3 - 2 + 0 - 8 + 5 \rightarrow -2 \rightarrow -2 + 11 \rightarrow 9 \\ 34567 & \rightarrow & 7 - 6 + 5 - 4 + 3 \rightarrow 5 \\ \times 12 & \rightarrow & 2 - 1 \rightarrow 1 \\ \hline 414804 & \rightarrow & 4 - 0 + 8 - 4 + 1 - 4 \rightarrow 5 \end{array}$$

$$\begin{array}{rcl}
56784 & \rightarrow 4-8+7-6+5 & \rightarrow 2 \\
-34567 & \rightarrow 7-6+5-4+3 & \rightarrow -5 \quad 2-5=-3 \\
\hline 22217 & \rightarrow 7-1+2-2+2 & \rightarrow 8 \quad -3+11=8 \\
1728 & \rightarrow 8-2+7-1 & \rightarrow 12 \rightarrow 2-1 \rightarrow 1 \\
\div 12 & \rightarrow 2-1 & \rightarrow 1 \rightarrow 1 \rightarrow 1 \\
\hline 144 & \rightarrow 4-4+1 & \rightarrow 1 \rightarrow 1 \rightarrow 1
\end{array}$$

(c) Use of Complementary Digits.

We use complementary digits 1, 2, 3, 4, 5 in the following sense that $a\bar{b}\bar{c}\bar{d}\bar{e}$ stands for $a \times 10^4 - b \times 10^3 - c \times 10^2 + d \times 10^1 - e \times 10^0$.

Thus

$$3\bar{4}\bar{5}\bar{6} = 3000 - 400 + 50 - 6$$

$$\bar{1}\bar{2}34 = -1000 + 200 + 30 + 4$$

All numbers positive and negative can be expressed in terms of the following digits :

$$0, 1, 2, 3, 4, 5, \bar{1}, \bar{2}, \bar{3}, \bar{4}$$

$$9786 = 10000 - 200 - 10 - 4 = 1\bar{2}\bar{1}\bar{4}$$

$$6781 = 10000 - 3000 - 200 - 20 + 1 = 1\bar{3}\bar{2}\bar{2}\bar{1}$$

$$-6781 = -10000 + 3000 + 200 + 20 - 1 = \bar{1}\bar{3}\bar{2}\bar{2}\bar{1}$$

$$-1111 = \bar{1}\bar{1}\bar{1}\bar{1}$$

Let the children express a number of positive or negative numbers in terms of these ten digits. Let them do addition problems in the notation e.g.,

$$\begin{array}{r}
8786 \\
+1\bar{3}\bar{4}\bar{5} \\
\hline
9441
\end{array}$$

To subtract a number, we can take the number with complementary digits and add. Thus to find $8786 - 1\bar{3}\bar{4}\bar{5}$, we find

$$\begin{array}{r}
8786 \\
+1\bar{3}\bar{4}\bar{5} \\
\hline
79\bar{1}\bar{1}
\end{array}$$

We can use this for checking results e.g.,

$$\begin{array}{r}
8435 \\
99 \\
\hline
75915 \\
75915 \\
\hline
835065
\end{array}
\quad
\begin{array}{r}
8435 \\
10\bar{1} \\
\hline
8435 \\
8435 \\
\hline
845135
\end{array}
\quad
\begin{array}{r}
840100 \\
-5035 \\
\hline
835065
\end{array}$$

Experiment 296. Patterns in Numerals.

(a) Number	Egyptian Symbol	Roman Symbols
1	1	I
5	11111	V
10	∩	X
50	∩∩∩∩∩	L
100	?	C
500	?????	D
1000	△ ↓ △	M

Let the children write numbers like 1256, 456, 1947, 1968 etc. in both Egyptian and Roman symbols and let them read some numbers in Egyptian and Roman forms.

(b) Let the children prepare the following chart :

	0	1	2	3	4	5	6	7	8	9	10	15	20
Hindu-Arabic													
Abacus													
Roman		I	II	III	IV	V	VI	VII	VIII	IX	X	XV	XX
Egyptian		1	11	111	1111	11111	111111	1111111	11111111	111111111	1111111111	11111111111	111111111111
Greek		α	β	γ	δ	ε	ς	ζ	η	θ	ι	ιε	χ
Mayan		.	..	...									
Brahmi (India 300 B.C.)		—	=	≡	Υ	ρ	6	7	~	?			
Arabic		•	ۛ	ۛۛ	ۛۛۛ	ۛۛۛۛ	ۛۛۛۛۛ	ۛۛۛۛۛۛ	ۛۛۛۛۛۛۛ	ۛۛۛۛۛۛۛۛ	ۛۛۛۛۛۛۛۛۛ	ۛۛۛۛۛۛۛۛۛۛ	ۛۛۛۛۛۛۛۛۛۛۛ
Devnagari		०	१	२	३	४	५	६	७	८	९		
Urdu		۰	۱	۲	۳	۴	۵	۶	۷	۸	۹		
Binary	0	1	10	11	100	101	110	111	1000	1001			
Ternary	0	1	2	10	11	12	20	21	22	100			

(c) Let the children try to do some arithmetic problems of addition and multiplication in Greek, Roman or Egyptian notation and see the advantages of the place value system introduced by ancient Indians.

Experiments : 297 to 300

Experiment 297. Lives of famous mathematicians of ancient times.

Children love stories and they will be interested in knowing about great mathematicians of ancient and modern times. The lives of great mathematicians can be used to illustrate many mathematical and scientific results and provide inspiration to the children and motivation for learning mathematics. New concepts can be introduced to receptive children when talking about some mathematician. Some of the incidents in their lives may create an impact on the minds of children which may last them their whole life-time.

Thales (624 to 548 B.C.)

He was one of the seven wise men of Greece and has been described as the first man to be called worthy of the title 'man of science'. He predicted the eclipse of the sun in 585 B.C. and thus created peace between warring tribes. He also measured the heights of Egyptian pyramids by using mathematics. There is also the story of his mule who used to roil in a stream of water to lighten his load of salt. Thales taught him a lesson by loading him with sponges. Thales also found many interesting results in geometry. Tell the children about eclipses and pyramids.

Pythagoras (572-507 B.C.)

He was a student of Thales and discovered the theorem that the square of the length of the hypotenuse of a right angled triangle is equal to the sum of squares of the lengths of the other two sides (see experiment 250). This theorem was however discovered earlier in India. His followers formed a secret society and all results found by them were attributed to Pythagoras. The Pythagoreans believed that odd numbers were divine and male and even numbers were earthly and female. They discovered the five regular solids (see experiment 276) and believed that these represent the elements of the universe : tetrahedron fire, cube-earth, octahedron-air, icosahedron-water and dodecahedron—the heavenly sphere. They also discovered the properties of triangular numbers, (see experiment 146). They also studied mathematics of music and by studying the shadow cast by the earth on the moon deduced that the earth must be a sphere.

Zeno (About 450 B.C.)

Zeno convinced many that a hare could not overtake a tortoise if the tortoise was given a headstart.

A B C D E

Suppose initially the hare is at A and tortoise is at B. By the time hare reaches B, tortoise is at C and by the time hare reaches C, tortoise is at D and so on, so that tortoise is always ahead of the hare. Can the children find an answer? The answer requires higher mathematics.

Plato (430 B.C.—349 B.C.)

It is said that at the entrance of his Academy was written : "Let no man ignorant of geometry enter here." This shows his great respect for mathematics. He insisted that in all geometrical constructions, only ruler and compasses should be used. There were three constructions which could not be done by ruler and compasses only, viz.,

- (i) Trisection of a given angle i.e., its division into 3 equal parts.
- (ii) Construction of a square equal in area to a given circle.
- (iii) Construction of a cube twice the size of the given cube.

For 2000 years, people tried to do these constructions with ruler and compasses only and failed. However recently it has been proved mathematically that these constructions are impossible, if only ruler and compasses are to be used. Children will study the proof when they grow up and learn higher mathematics. Plato believed that

mathematics should be developed primarily for its own sake and only secondarily for its application. He wanted that in its teaching, learning and amusement should be combined.

Euclid (About 365 B.C.)

Euclid collected and systematised all known results in geometry in his famous book known as *Euclid's Elements* which consists of thirteen scrolls. The first six of these included the properties of triangles, parallelograms, circles, polygons, etc. He filled in gaps in the theorems and added problems. His book has had a great influence on the development of mathematics and science and is one of the few books which continue to be taught even after more than 2000 years of its having been written.

Archimedes (287 B.C.—212 B.C.)

He was the greatest mathematician of ancient times and he is considered to be one of the three greatest of all times. Unlike other Greek mathematicians, he made practical use of mathematics in war. The story is told of how the king ordered him to find whether a crown made for him was of pure gold or whether it consisted of some cheaper metal also. He thought and thought over it and ultimately found the solution when he was taking a bath. He was so excited that he ran down the street shouting 'Eureka, Eureka', which means 'I have found it'. The children will study the principle he discovered in their science course. Archimedes also discovered the principle of levers and he is reputed to have said : "Give me a place to stand and I can use a lever to move the earth." Archimedes found the value of π (the ratio of circumference of a circle to its diameter) correct to two decimal places. In India it was found much more accurately later. Recently, the computers have been used to find its value correct even to 2000 decimal places. If possible, tell the children about Archimedes principle, levers and computers. Archimedes was killed by a soldier when he was engrossed in solving a mathematical problem. He told the soldier : 'Go away. Don't spoil my circles.' Good mathematics can be done when people get so engrossed with it.

Eratosthenes (278 B.C.—194 B.C.)

Eratosthenes measured the circumference of the Earth by using a mathematical formula and his value of 25,000 miles was almost correct. He gave the 'sieve' method of finding prime numbers (see experiment 60).

Appollonius (About 225 B.C.)

He is famous for studying the properties of conics. These are curves obtained when a cone is cut by planes perpendicular to the axis (giving a circle) inclined to the axis (giving an ellipse), parallel to the axis (giving a hyperbola), parallel to a generator (giving a parabola). The sections can be shown to the children. They can also be seen on a well when a light from a torch is thrown on it at various angles. These curves were studied because they were interesting. More than fifteen hundred years later, it was shown that the various planets move about the sun in paths which are ellipses. The satellites also move round the earth in elliptic orbits.

Indian Mathematicians.

Indian mathematicians made important contributions to mathematics in this period though no names are known, because it was not the custom in India to write names. In arithmetic, the Indians gave the decimal system, the place value system and zero. The zero existed in India before 200 B.C. Children can realize the importance of the place value system when they find how complicated it is to do arithmetic in Roman or Greek or Egyptian systems. Laplace wrote : "The idea of expressing all quantities by nine figures (or digits) whereby is imparted to them both an absolute value and a value by position is so simple that this very simplicity is the reason for our not being sufficiently aware how much admiration it deserves." Prof. Halstead wrote recently : "The importance of the creation of zero mark can never be exaggerated. This giving to airy nothing, not merely a local habitation and a name, a picture, a symbol but helpful power, is the characteristic of the Hindu race whence it sprang. It is like coining the Nirvana into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power."

In geometry, in the seven *Sulva-sutras* written in the period 800 B.C.—500 B.C. many important results given later by Greeks are given including the theorem known after Pythagoras but which should correctly be called "*Sulva Sutra Theorem*". Pythagoras gave the general solution $a=2n+1$, $b=2n^2+2n$, $c=2n^2+2n+1$ of the equation $a^2+b^2=c^2$. The last two numbers differ by unity. The *Sulva Sutra*s mentioned solutions (3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17), (12, 35, 37), $(5\sqrt{3}, 12\sqrt{3}, 13\sqrt{3})$, $(15\sqrt{2}, 36\sqrt{2}, 39\sqrt{2})$ etc. Later Brahmagupta (628 A.D.) gave the general solution $m^2-n^2, 2mn, m^2+n^2$. The *Sulva Sutra*s also showed a knowledge of irrational numbers two or

three centuries before Pythagoreans could who discovered that $\sqrt{2}$ was irrational, but were afraid to disclose this fact to others.

The Jainas treated mathematics as a part of their religion and gave many results in arithmetic and geometry. They gave $\pi = \sqrt{10}$ in 500 B.C. They also discussed permutations and laws of indices.

According to a Russian author "Geometry was created in India". He also gives the proof of the formula for the area of the circle as having been found by an Indian mathematician Ganesa in 3000 B.C. There was no mathematician of this name in India. This reference is perhaps to lord Ganesa. It states that this proof is given in a temple, but it is not known which temple is being referred to.

Experiment 298. Lives of Famous Mathematicians of Middle Period.

Diophantus (About 275 A.D.)

He was one of first Western mathematicians to use symbols. Diophantus was concerned with the determination of rational solutions of the equation $by - ax = c$ and also particular cases of equation of the form $y^2 = ax^2 + c$.

The problem of solving the equation $by - ax = c$ for x, y as positive integers when a, b, c are given integers is called Diophantus problem, though more properly it should be called Arya Bhatta or Kuttaka problems as this problem in this form was first considered by Arya Bhatta (499 A.D.) by a method known as Kuttake.

Arya Bhatta (About 500 A.D.)

In his famous book Arya Bhatteya, he gave methods for finding square root and cube root and stated the value of π to be 3.1416. He gave tables for sines and cosines and gave the method of solving the equations known after Diophantus.

Brahmgupta (about 630 A.D.)

He gave a method for solving the indeterminate equation $Nx^2 + 1 = y^2$, a general formula $a^2 - 2mn, b^2 - m^2 - n^2, c^2 = m^2 + n^2$ for solving $a^2 + b^2 = c^2$, results for cyclic quadrilaterals and was the first mathematician to give interpolation formulae using second order differences. Bhaskara called him Ganita Chakra Chudamani i.e. gem of the circle of mathematicians and he richly deserved the title.

Al-Khowarizmi (About 820 A.D.)

He was an Arab scholar who used Hindu numerals to solve equations. A corruption of his name became "algorism". For a long time algorism meant reckoning with Hindu-Arabic numerals.

Mahaviracharya (About 850 A.D.)

He showed that a number times 0 is 0. An important feature of his treatment of division of fractions was his rule to invert the divisor and multiply. He gave the following 'garland' products which resemble a garland since these gave same numbers read from left to right or right to left. They will be of interest to the children.

$$139 \times 129 = 15151$$

$$27994681 \times 441 = 1234565432$$

$$12345679 \times 9 = 111111111$$

$$333333666617 \times 33 = 110000111000011$$

$$14287143 \times 7 = 100010001$$

$$142857143 \times 7 = 1000000001$$

$$152207 \times 73 = 111111111$$

$$11011011 \times 91 = 1002002001$$

He solved problems on quadratic equations, gave the general formulae for $\sqrt{Cr}$, expressed any fraction as the sum of unit fractions, solved some problems for cyclic quadrilaterals and was the first Indian mathematician to refer to the ellipse.

Omar Khayyam (About 100 A.D.)

He was a great Persian poet famous for his Rubaiyats. He wrote the best book on algebra produced by the Persians

Bhaskaracharya (About 1150 A.D.)

Bhaskaracharya was the author of the Siddhanta Shiromani which is divided into four parts namely Leelavati, Beejganita, Goladhyaya and Grahaganita. The first dealt with arithmetic and geometry, the second with algebra and the last two deal with astronomy. He derived inspiration from the work of other Hindu mathematicians like Sridhara, Padamnabha, Brahmgupta and Mahaviracharya, but he systematised their results and added his own contributions. There is a story that Bhaskaracharya had a daughter by the name of Leelavati and Bhaskaracharya came to know from her horoscope that her married life would be cut short unless her marriage was performed at a particular moment. He made a sand glass which was a device in

which sand flows from one vessel to another through a small orifice in a fixed interval of time. However as luck would have it, on the day before the marriage, Leelavati peeped into the vessel and a stone from her nose ring fell into the vessel and retarded the flow of sand and so she could not be married at the auspicious moment. To console her Bhaskara taught her arithmetic and named the first part after her name. There is however no basis for this story and it may not be correct. Leelavati starts with a prayer to Lord Ganesa and then has the following sections : the number systems, the eight operations (additions, subtraction, multiplication, division, square, cube, square root, cube root), fractions, zero, rule of three, mixture, interest, progressions, geometry, mensuration, kuttaka and permutations. Bhaskara found the area and volume of a sphere by method of summation used in integral calculus. He also gave the concept of instantaneous motion and initiated the study of differential calculus and suggested that the differential coefficient vanishes at an extreme value of the function. Differential calculus was given a good beginning by Bhaskara, but it was not developed in India later and was re-discovered more completely by Newton and Leibnitz. Bhaskara also made important contributions to trigonometry.

Fibonacci (About 1200 A.D.)

The well-known Fibonacci series is known after him. He was also called Leonardo Pisano.

Experiment. 299. Lives of some Famous European Mathematicians.

Copernicus (1473—1543 A.D.)

He was a famous Polish astronomer and mathematician. He first gave the theory that the earth moves round the sun and that the moon merely reflects light received from the sun. He also wrote a book on trigonometry as a help in the study of astronomy. His book on astronomy was published in 1543, the year of his death but the book was banned for a number of years, as religious bodies were against it.

Vieta (1540—1603 A.D.)

He was among the first in Europe to use letter to represent numbers in algebra and often used vowels for unknowns and consonants for constants.

Napier (1550—1617 A.D.)

He invented Napierian logarithms *i.e.* logarithms to base e which the children will read later. He also invented Napier rods which the children have read about.

Briggs (1561—1631 A.D.)

He invented the common logarithms to base 10 which are useful for multiplication and division problems.

Galelio (1564—1642 A.D.)

He was an Italian mathematician who found the time of oscillation of a simple pendulum by considering the way in which the cathedral lamp in his town swung to and fro and he used his pulse beats to time it. He started the study of dynamics by throwing stones from the leaning towers of Pisa. He showed that the speed of a falling body increases by 32 feet per second. This was against the Greek theory that heavier bodies fall more quickly than lighter bodies and he had to resign his university position because of this discovery. He made a telescope and his observations supported Copernician theory. He discovered the four moons which orbit around Jupiter, the largest of all planets.

Kepler (1571—1630 A.D.)

On the basis of his and earlier observations of Tycho Brahe, Kepler deduced his famous laws of planetary motion according to one of which planets move round the Sun in elliptic and not circular orbits.

Descartes (1596—1650 A.D.)

He is known as the father of modern philosophy and of analytic geometry. He said, "I think, therefore I exist." Cartesian coordinates are named after him.

Fermat (1601 to 1665 A.D.)

He was the greatest mathematician of the seventeenth century and is called the prince of amateur mathematicians since he worked as a king's councillor and not as a professional mathematician. He stated many theorems without proof which were later proved by others *e.g.* he stated that every prime number of the form $4n+1$ can be expressed as the sum of two squares in one and only one way and that $n^p - n$ is divisible by p where p is a prime (let the children verify these). He also stated that he had found a marvellous proof

that $x^n + y^n = z^n$ has no solution in integers when $n > 2$ (let the children try to find a solution). This theorem known as Fermat's theorem has not been proved till today.

Pascal (1623—1662 A.D.)

When young he was weak. His father advised him, therefore, not to tax his brain with mathematics. This aroused the curiosity of the child and he began to study geometry. At the age of 11, he had studied the first 32 propositions of Euclid and at the age of 16, he had written a complete treatise on conics. He also found the Pascal triangle while solving a problem on probability. The triangle was however known to Jain mathematicians in India about 1000 years earlier.

Newton (1642—1727 A.D.)

He is considered as one of the greatest, if not the greatest, mathematicians of all times. He was one of the inventors of calculus and made fundamental contributions to mechanics, gravitation, optics, planetary motions etc. He discovered calculus, which he called fluxions in 1666, but did not publish his findings till 1687. In the mean time Leibniz, a German mathematician, made an identical discovery in 1684. There was a bitter dispute as to who discovered this important subject first. It is now agreed that both obtained their results independently.

Leibniz (1646—1716 A.D.)

He is one of the founders of calculus. He also invented the binary system used in computers and dreamt of mathematical logic as a school boy. He was also a great philosopher.

Euler (1707—1783 A.D.)

He was a Swiss mathematician and he helped the Czarina of Russia in suppressing Dideriot's irreligious views by using mathematics. Dideriot used to say "God does not exist". Euler said $\frac{a+bx^n}{n} = x$, hence God exists. Dideriot did not know mathematics and so he admitted defeat. He wrote thousands of pages of mathematics and he created a good deal of mathematics even after he had become almost totally blind.

Lagrange (1736—1813 A.D.)

He was a French mathematician and studied dynamics, hydrodynamics, calculus of variations etc.

Laplace (1749—1827 A.D.)

He was a French mathematician who was educated by those who recognized his ability. His well-known equation plays an important role in mathematics.

Fourier (1768—1830 A.D.)

He was a French mathematician who developed Fourier series to study problems of heat conduction.

Gauss (1777—1855 A.D.)

He is considered, along with Archimedes and Newton, as one of the three greatest mathematicians of all times. He was the first to anticipate a non-Euclidean geometry, though he did not publish it. He could do complicated problems at the age of 3 and he surprised his teacher by finding the sum of numbers from 1 to 100 in a few seconds.

Cauchy (1789—1857 A.D.)

He was a French mathematician who made fundamental contributions to complex analysis.

Abel (1802—1829 A.D.)

Though he died at the early age of 27, he did very good work in Algebra and Analysis. He proved that a general equation of the fifth degree could not be solved in the same sense in which equations of lower degrees could be solved.

Moebius (1790—1869 A.D.)

He was the first to introduce homogeneous coordinates which are accepted tools for algebraic methods of algebraic geometry. He is considered as one of the founders of modern science of topology. The Moebius strip, a surface with only one side is easy to make and the children like to study its properties.

Lobachevsky (1793—1856 A.D.)

This Russian mathematician was perhaps the first to publish a really systematic treatment of non-Euclidean geometry.

Bolyai (1802—1860 A.D.)

He also constructed a non-Euclidean geometry and published it as an appendix to his father's book.

Hamilton (1805—1865 A.D.)

He was an Irish mathematician who discovered quaternions. He perceived that an algebra where multiplication was not commutative was possible. He brooded over the problem for many years and he suddenly conceived of it when walking along a canal. He immediately took out his pen knife and carved there $i^2=j^2=k^2=ijk=-1$ which was the basis of his systems of quaternions.

Experiment 300. Some Mathematicians of the Last Century.

Weirstrass (1815—1897 A.D.)

He was a German mathematician who laid the foundations of a rigorous study of calculus.

Cayley (1815—1897 A.D.)

He was an English mathematician who made important contributions to matrices.

Boole (1815—1864 A.D.)

He was an English mathematician who laid the foundations of modern mathematical theory of logic in his book 'laws of thought'.

Reimann (1826—1866 A.D.)

He laid the foundations of Reimannian geometry which was later used by Einstein in his theory of relativity.

Dedekind (1831—1916 A.D.)

He was a German mathematician who gave a rigorous theory of real numbers through his concept of cuts.

Poincare (1845—1912 A.D.)

He was a French mathematician who contributed greatly to mechanics, probability and topology. He is believed to be the last all-rounder in mathematics.

Cantor (1845—1918 A.D.)

He was a German mathematician who created the fundamental theory of sets. His theory made it as easy for mathematicians to deal with infinite numbers as they had found it earlier to deal with finite numbers.

Kovalusky (1850—1935 A.D.)

She was a Russian mathematician who made important contributions to differential equations.

Peano (1858—1932 A.D.)

He was an Italian mathematician who is considered as one of the creators of modern logic. He gave axiomatic treatment of the natural number system.

Hilbert (1862—1943 A.D.)

He was a German mathematician who made a deep study of foundations of geometry and was the leader of the formalist school of mathematics.

Einstein (1879—1955 A.D.)

He invented the theory of relativity that challenged Newton's mechanics after 100 years in some respects. He was a dull student and at the age of 16 was expelled from school for poor marks. After 22, he did such good work that he is considered one of the greatest mathematicians and physicist of all times.

Ramanujan (1887—1920 A.D.)

He was the greatest mathematical genius whom India has produced in recent times. He re-discovered for himself the work of three centuries of mathematics in Europe. Later he was discovered by Prof. Hardy of Cambridge. He died at an early age, still he did very good work.

Von Neumann (1903—1957 A.D.)

He was a Hungarian mathematician who made important contributions to quantum mechanics and operator theory. He was one of the inventors of the theory of games which is used in economics and war. He also made important contributions to computers. He learnt college calculus at the age of 8.

Weiner (1895—1964 A.D.)

He founded the science of cybernetics which is science of control in man and machines. He was also a child prodigy.

EXERCISES

1.* Find the pairs of consecutive members (<1000) whose sum is a square number. Discover the nature of such square numbers.

$$\begin{array}{ll} 4+5=9 & 12+13=25 \\ 24+25=49 & 40+41=81 \\ 60+61=121 & 84+85=169 \\ \dots\dots\dots & \dots\dots\dots \end{array}$$

2. Find the pairs of consecutive odd numbers (<1000) whose sum is a square number. Discover the nature of such square numbers.

$$\begin{array}{ll} 1+3=4 & 7+9=16 \\ 17+19=36 & 31+33=64 \\ \dots\dots\dots & \dots\dots\dots \end{array}$$

3. Write numbers from 1 to 1000 and draw circles round the pairs of exercise 1 and discover if there is any pattern in the number of numbers left in the gaps.

1, 2, 3, (4), (5), 6, 7, 8, 9, 10, 11, (12), (13), 14, 15, 16, 17, 18, 19, 20,
21, 22, 23, (24), (25), 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37,
38, 39, (40), (41), 42, 43, 44, 45, 46, 47, 48, 49, 50.....

4. Write odd numbers from 1 to 1000 and draw circles round the pairs of exercise 2 and discover if there is any pattern in the number of numbers left in the gaps.

(1), (3), 5, (7), (9), 11, 13, 15, (17),
(19), 21, 23, 25, 27, 29, (31), (33).....

*The first ten exercises have been suggested by
Shri P.K. Srinivasan of Kanpur Study Group.

5. Find out if there are any pairs of consecutive even numbers whose sum is a square number.

6. Consider the Pythagorean triplets

(3, 4, 5), (5, 12, 13) (6, 8, 10) (7, 24, 25), (8, 15, 17),
(9, 40, 41),.....

Note that

$$4+5=3^2, 12+13=5^2, 8+10=\frac{6^2}{2}, 24+25=7^2, 15+17=\frac{8^2}{2}, 40+41=9^2, \dots\dots$$

Comment on the pattern.

7. Assume incomes for A and B. Express A's as such more/less than B's and B's as so much less/more than A's. Tabulate as follows :

A	B
Rs. 2000	Rs. 3000
$-\frac{1}{3}$	$+\frac{1}{2}$

This means that A gets $\frac{1}{3}$ less than what B gets and B gets $\frac{1}{2}$ more than what A gets.

Show also that

A	B	A	B
$-\frac{1}{3}$	$+\frac{1}{2}$	$-\frac{2}{7}$	$+\frac{2}{5}$
$-\frac{1}{4}$	$+\frac{1}{3}$	$-\frac{3}{11}$	$+\frac{3}{8}$
$-\frac{1}{5}$	$+\frac{1}{4}$	$-\frac{4}{13}$	$+\frac{4}{9}$
$-\frac{1}{6}$	$+\frac{1}{5}$		

Discover the pattern.

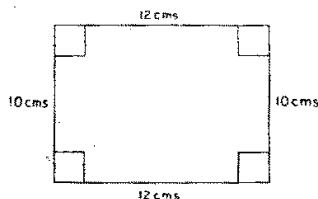
8. Using only one and two paise coins, find the number of different ways in which one can pay amounts of one paise, two paise, three paise only and show that these numbers form a Fibonacci sequence.

9. Continue the following rational approximations to $\sqrt{2}$ to get the value of $\sqrt{2}$ correct to four decimal places.

$$\begin{array}{lcl}
 1 \times 1 = 1 & 1.1 \times 1.1 = 1.21 & 1.41 \times 1.41 = 1.9881 \\
 & 1.2 \times 1.2 = 1.44 & 1.42 \times 1.42 = 2.0164 \\
 & 1.3 \times 1.3 = 1.69 & \\
 2 \times 2 = 4 & 1.4 \times 1.4 = 1.96 & \\
 & 1.5 \times 1.5 = 2.25 &
 \end{array}$$

Do the same for $\sqrt{3}$ and $\sqrt{5}$.

10. Study the pattern in the capacities of rectangular boxes made from a sheet of 12 cm. $\times$ 10 cm. when square corners of different sides are removed and the sides are folded.



When is the capacity maximum?

Side of square corner in cm.	Box dimensions in cm.	Capacity in c.c.
$\frac{1}{2}$	11, 9, $\frac{1}{2}$	$49\frac{1}{2}$
1	10, 8, 1	80
$1\frac{1}{2}$	9, 7, $1\frac{1}{2}$	$94\frac{1}{2}$
2	8, 6, 2	96
$2\frac{1}{2}$	7, 5, $2\frac{1}{2}$	$87\frac{1}{2}$
3	6, 4, 3	72

11. Factorise $x^n - 1$ for various values of n and show that the absolute value of each coefficient in each factor does not exceed unity, upto $n=104$, but that for $n=105$, $x^{105} - 1$ has a factor.

$x^{49} + x^{47} + x^{46} - x^{43} - x^{42} - 2x^{41} - x^{40} - x^{39} + x^{36} + x^{35} + x^{34} + x^{33} + x^{32} + x^{31} - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} - x^9 - x^8 - 2x^7 - x^6 - x^5 + x^2 + x + 1$ in which the absolute value of the coefficients of each of x^7 and x^{41} is 2.

*12. Each of the prime numbers of the form $4n+1$ (i.e., numbers 5, 13, 17, 29, 37, 41,.....) can be expressed as the sum of exactly two square numbers, while each one of the prime numbers of the form $4n+3$ (i.e., numbers 3, 7, 11, 19, 23, 31,.....) can not be so expressed. Verify the result for all prime numbers less than 1000.

$$5 = 1^2 + 2^2, \quad 13 = 2^2 + 3^2, \quad 17 = 1^2 + 4^2, \quad 29 = 2^2 + 5^2, \quad 37 = 1^2 + 6^2, \quad 41 = 4^2 + 5^2, \quad 53 = 2^2 + 7^2, \dots$$

**13. Prepare a table of prime numbers less than 500 in the following manner:

Form	Prime Numbers
(i) $4n+1$	5, 13, 17, 29, 37, 41, 53, 61,.....
(ii) $8n+1$	17, 41, 73, 89, 97,
(iii) $8n+3$	43, 59, 67, 83,.....
(iv) $8n-1$	7, 23, 31, 47, 71, 79,
(v) $6n+1$	7, 13, 19, 37, 43, 61, 67, 73, 79, 97,.....
(vi) $12n+1$	13, 37, 61, 73, 97,.....
(vii) $20n+1$	41, 61, 101,.....
(viii) $20n+9$	29, 89,.....
(ix) $10n+1$	11, 31, 41, 61, 71,.....
(x) $10n+9$	19, 29, 59, 79, 89,.....
(xi) $14n+1$	29, 43, 71,.....
(xii) $14n+9$	23, 37, 79,.....
(xiii) $14n+25$	53, 67,.....

Verify that each number of (i) can be expressed as $x^2 + y^2$, each number of (ii) and (iii) as $x^2 + 2y^2$, each number of (iv) as $x^2 - 2y^2$, each number of (v) as $x^2 + 3y^2$, each number of (vi) as $x^2 - 3y^2$, each number of (vii) and (viii) as $x^2 + 5y^2$, each number of (ix) and (x) as $x^2 - 5y^2$, each number of (xi), (xii) and (xiii) by $x^2 + 7y^2$ and so on.

14. Verify that the following are solutions of $x^3 + y^3 + z^3 = u^3$ (1, 6, 8, 9), (3, 4, 5, 6), (18, 19, 21, 28), (7, 14, 17, 20), (19, 60, 69, 82), (15, 82, 89, 108), (3, 36, 37, 46), (1, 135, 138, 172).

Find two more such sets.

*This result is known as Fermat's theorem.

**Exercises 13—20 are based on Ramanujan's note-books.

15. Verify that the following are solutions of

$$x^4 + y^4 + z^4 + u^4 + v^4 = w^4$$

(4, 6, 8, 9, 14, 15), (1, 2, 12, 24, 44, 45), (4, 21, 22, 26, 28, 35),
(4, 8, 13, 28, 54, 55), (1, 8, 12, 32, 64, 65), (22, 28, 63, 72, 94, 105),
(22, 52, 57, 74, 76, 95), (2, 39, 44, 46, 52, 65).

Find two more such sets.

16. Verify that the following are solutions of

$$x^5 + y^5 + z^5 + u^5 + v^5 + w^5 = t^5$$

(4, 5, 6, 7, 9, 11, 12), (5, 10, 11, 16, 19, 21).

Find one more such set.

17. Verify that the following are solutions of

$$x^4 + y^4 + z^4 = u^4 + v^4 + w^4$$

(2, 4, 7, 3, 6, 6), (3, 7, 8, 1, 2, 9), (6, 9, 12, 2, 2, 13).

Find one more such set.

18. Verify that the following are solutions of

$$x^4 + y^4 + z^4 = u^4 + v^4 + w^4 + t^4$$

(2, 2, 7, 4, 4, 5, 6), (3, 9, 14, 7, 8, 13), (7, 10, 13, 5, 5, 6, 14).

Find one more such set.

19. Show that the smallest number which can be expressed as the sum of two cubes in 2 ways is 1729. Verify also

$$\begin{aligned} 1729 &= 10^3 + 9^3 = 12^3 + 1^3; & 4104 &= 16^3 + 2^3 = 15^3 + 9^3 \\ 13832 &= 24^3 + 2^3 = 20^3 + 18^3; & 40033 &= 34^3 + 9^3 = 33^3 + 16^3 \\ 64232 &= 39^3 + 17^3 = 36^3 + 26^3; & 110808 &= 48^3 + 6^3 = 45^3 + 27^3 \\ 635,318,657 &= 134^3 + 134^3 = 59^3 + 158^3. \end{aligned}$$

Express 842751 as the sum of two cubes in two ways.

20. Verify that the following satisfy the pattern

$$a + b + c + \dots = p + q + r \dots$$

and $a^p \cdot b^q \cdot c^r \dots = p^p q^q r^r \dots$

$$\begin{aligned} 1^1 \cdot 1^1 \cdot 2^2 \cdot 6^6 &= 3^3 \cdot 3^3 \cdot 4^4 \\ 1^1 \cdot 1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdot 5^5 \cdot 6^6 &= 3^3 \cdot 3^3 \cdot 3^3 \cdot 4^4 \cdot 4^4 \cdot 5^5 \\ 1^1 \cdot 8^8 \cdot 9^9 &= 3^3 \cdot 3^3 \cdot 12^{12} \\ 1^1 \cdot 3^3 \cdot 12^{12} \cdot 20^{20} &= 15^{15} \cdot 16^{16} \\ 1^1 \cdot 4^4 \cdot 20^{20} \cdot 30^{30} &= 6^6 \cdot 24^{24} \cdot 25^{25}. \end{aligned}$$

Find one more such example.

*21. Verify the following :

$$(i) 161^2 + 162^2 + 163^2 + 164^2 + 165^2 + 166^2 + 167^2 + 168^2 + 169^2 + 170^2 + 171^2 + 172^2 + 173^2 + 174^2 + 175^2 = 243^2 + 244^2 + 245^2 + 246^2 + 247^2 + 248^2 + 249^2$$

$$(ii) 236^2 + 237^2 + 238^2 + 239^2 + 240^2 + 241^2 + 242^2 + 243^2 + 244^2 = 318^2 + 320^2 + 322^2 + 324^2 + 326^2$$

$$(iii) 174 \times 175 + 175 \times 176 + 176 \times 177 + 177 \times 178 + 178 \times 179 + 179 \times 180 + 180 \times 181 + 181 \times 182 = 503 \times 504.$$

**22. Multiply 12345678987654321 by 108, 162, 207, 225, 243, 405 and comment on the results.

23. Multiply 207 by 121, 12321, 1234321, 123454321, 12345654321, 1234567654321, 123456787654321, 12345678987654321 and comment on the results.

***24. Verify

$$\begin{aligned} 4556 &= 35 \times 53 + 37 \times 73 = 26 \times 62 + 46 \times 64 \\ 5040 &= 15 \times 51 + 57 \times 75 = 24 \times 42 + 48 \times 84 \\ 6328 &= 37 \times 73 + 39 \times 93 = 28 \times 82 + 48 \times 84 \\ 5239 &= 17 \times 71 + 48 \times 84 = 39 \times 93 + 26 \times 62 \end{aligned}$$

Observe the pattern and express 2620, 3908, 6976, 7460, 9880, 3709, 4151, 6613, 8549 in the same way.

25. List all the divisors of all composite numbers less than 500. Also factorise each number into prime factors. Factorise also the number of divisors. Give the relation between the number of divisors and the powers of the prime factors.

$$360 = 2^3 \cdot 3^2 \cdot 5 \text{ has } 24 = (3+1)(2+1)(1+1) \text{ divisors.}$$

26. A number is said to be perfect if the sum of its divisors is twice the number. Show that 6 and 28 are perfect numbers.

27. Consider the polygonal numbers represented by the figures at the next page.

*Patterns found by M. N. Khatri.

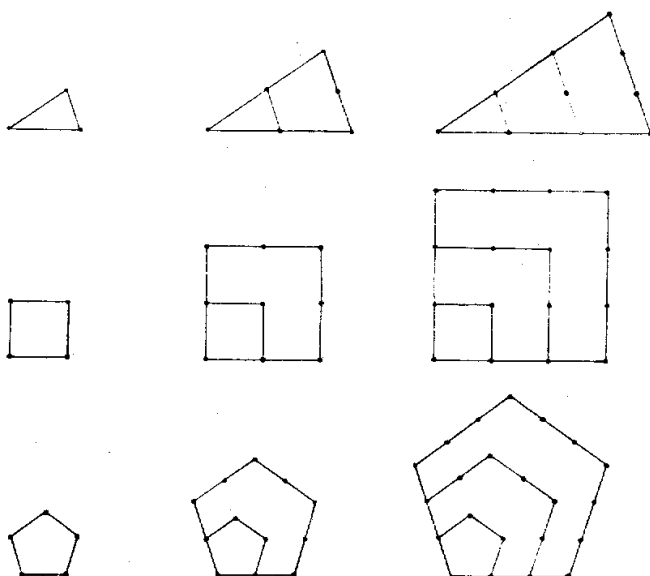
**Patterns found by D. R. Kshrekar.

***Patterns found by R.V. Iyer.

Rank 1 Rank 2

Rank 3

Rank 4



Figures in the first row are Triangular.
Figures in the second row are Square.
Figures in the third row are Pentagonal.

Prepare the following table :

Number	Rank											
	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th	12th
Triangular	1	3	6	10	15	21	28	36	45	55	66	78
Square	1	4	9	16	25	36	49	64	81	100	121	144
Pentagonal	1	5	12	22	35	51	70	92	117	145	176	210
Hexagonal	1	6	15	28	45	66	91	120	153	190	231	276
Heptagonal	1	7	18	34	55	81	112	148	189	235	286	342
Octagonal	1	8	21	40	65	96	133	176	225	280	341	408

Note the patterns in this table.

- (i) Differences between consecutive members of a row are in arithmetic progression and the common differences for different rows form an arithmetic progression.
- (ii) The numbers of each column are in arithmetical progression and the common differences between various columns are themselves in arithmetic progression.
- (iii) A square number of a certain rank is the sum of the triangular number of that rank and the triangular number of the preceding rank.
- (iv) Every octagonal number is equal to its rank plus the triangular number of preceding rank multiplied by 6.
- (v) Eight times a triangular number is a square number. Give geometrical interpretations of each and notice other similar patterns.

28. Show geometrically that

- (i) every hexagonal number is equal to its rank plus four times the triangular number of preceding rank.
- (ii) nine times of triangular number of rank n plus one is a triangular number of rank $3n+1$.

29. Show that

- (i) a pentagonal number cannot end in 3, 4, 8 or 9.
- (ii) a hexagonal number cannot end in 2, 4, 7 or 9.

30. Find two numbers, neither equal to unity, which are simultaneously square and triangular.

31. A figurate number of any order in three dimensions of a certain rank n is the sum of n figurate numbers of that order in two dimensions. In the same way, a figurate number of any order of four dimensions of a certain rank n is the sum of n figurate numbers of that order in three dimensions. Prepare tables for figurate numbers of different ranks in three and four dimensions. Give geometrical interpretations wherever possible. The table of question 27 gives figurate numbers of order 3, 4, 5, 6, 7, 8 of dimension two.

32. Show that all positive integers x which make $x(x+180)$ a square number are :

$$x=12, 16, 60, 144, 320, 588, 1936.$$

33. Represent in all possible ways 1547 and 1768 as a difference of two squares.

$$[1547 = 774^2 - 773^2 = 114^2 - 107^2 = 66^2 - 53^2 = 54^2 - 37^2 \\ 1768 = 443^2 - 441^2 = 223^2 - 219^2 = 47^2 - 21^2 = 43^2 - 9^2]$$

34. Two numbers are said to be 'amicable' if each equals the sum of proper divisors of the others (*i.e.* not considering the number itself as a divisor)

Show that 220 and 284 and 17296 and 28496 are amicable pairs.

35. A chain of numbers is said to be 'sociable' if each is the sum of the proper divisors of the preceding number, the last being considered as preceding the first number of the chain. Show that the following numbers form a 'sociable' chain.

$$14288, 15472, 14536, 14264, 12496.$$

36. Show that for $n=2, 3, 4, 6, 8, 12, 18, 24$ and 30 , the numbers less than n and relatively prime to n are unity and primes. Does this property hold for any number greater than 30 ? No.

37. 24 is a number divisible by all numbers not exceeding its square root. See if you can find a number greater than 24 with this property. (This is not possible.)

38. Verify (for $n < 20$) that $1 \times 2 \times 3 \times 4 \dots \times (n-1)$ is divisible by n if and only if n is prime (Wilson's Theorem).

39. Verify for some n and p that $n^{p-1} - 1$ is divisible by p if n is an integer not divisible by the prime number p (Fermat's theorem).

40. Verify (for $n < 100$) that every even number greater than 2 is a sum of two primes (*e.g.*, $30 = 13 + 17$).
(Goldbach's conjecture which is still unproved)

41. Verify (for $n < 100$) that every number n can be expressed as the sum of 4 or less squares or as the sum of 9 or less cubes.
(Waring's conjecture)

42. Consider the series of Fibonacci *viz.*
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1591, in which each term is the sum of the preceding two. Guess the relations, if any, between

(a) square of the n th term and

- (i) product of $(n-1)$ th and $(n+1)$ th terms
- (ii) product of $(n-2)$ th and $(n+2)$ th terms
- (iii) product of $(n-3)$ th and $(n+3)$ th terms
- (iv) product of $(n-4)$ th and $(n+4)$ th terms and so on.

(b) product of n th and $(n+1)$ th terms and

- (i) product of $(n-1)$ th and $(n+2)$ th terms
- (ii) product of $(n-2)$ th and $(n+3)$ th terms and so on.

(c) sum of squares of two consecutive terms and some term of the series.

(d) difference of squares of two consecutive terms and some term of the series.

(e) relation between any three consecutive terms.

(f) cube of a term + cube of the next term - cube of preceding term and some term of the Fibonacci series.

(g) difference of the $(n+1)$ th term of Fibonacci series and the product of the golden section $\frac{1}{2}(1 + \sqrt{5})$ with the n th term of the series, as n increases.

43. Verify the following :

$$3^4 + 4^4 + 5^4 = 5^2 + 19^2 + 24^2 \\ 3^8 + 4^8 + 5^8 = 5^4 + 19^4 + 24^4 \\ 7^2 + 34^2 + 41^2 = 14^2 + 29^2 + 43^2 \\ 7^4 + 34^4 + 41^4 = 14^4 + 29^4 + 43^4$$

What is the pattern here ?

44. (i) $\frac{1}{7} = .142857142857...$

Multiply 142857 successively by 1, 2, 3, 4, 5, 6, 7 and comment on the pattern.

(ii) $\frac{1}{17} = .05882352941176470588...$

Multiply 5882352941176470 by 1, 2, 3, 4, ... and comment on the pattern.

(iii) $\frac{1}{89} = .03448275862068965517241379310344827.....$

Can you multiply 3448275862068965517241379310 by any number from 1 to 28 at once ?

(iv) Get some more similar patterns.

45. (i) Make 1000 using only 8's ? $(8+8+8+88+888)$.
 (ii) Make 100 using only 7's ? $(7/7 \times 7/7 \times 7/7 \text{ or } \frac{7}{7/7})$.
 (iii) Make 20 using only two 3's $\frac{\angle 3}{.3}$.

46. (i) $ABCDE \times 4 = EDCBA$.

Find A, B, C, D, E ; [A=2, B=1, C=9, D=7, E=8.]

- (ii) Solve $\frac{PORK}{CHOP} = C, C > 2$ $\left[\frac{9867}{3289} = 3. \right]$

47. Show that below 10,000, there are only two four figure numbers which are integral multiples of their reversals. [These are 8712 and 9801.]

48. Show that there are just four numbers (after 1) which are the sums of the cubes of their digits viz.,

$$153 = 1^3 + 5^3 + 3^3 ; 370 = 3^3 + 7^3 + 0^3$$

$$371 = 3^3 + 7^3 + 1^3 ; 407 = 4^3 + 0^3 + 7^3.$$

49. Give multiplication sums in which each of the nine non-zero digits occur once only

$$[48 \times 159, 12 \times 483, 42 \times 138, 18 \times 297, 27 \times 198,$$

$$39 \times 186, 48 \times 159, 28 \times 157, 4 \times 1738, 4 \times 1963.]$$

50. $(12)^2 = 144, (21)^2 = 441, (13)^2 = 169, (31)^2 = 961$.

Find similar three digit numbers.

$$[(102)^2, (103)^2, (112)^2, (113)^2, (122)^2.]$$

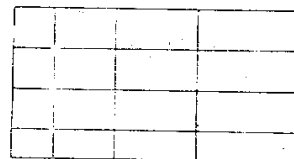
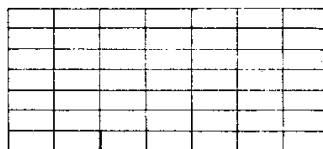
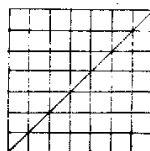
51. Just as polyminoes are formed from squares, the polyiamonds are figures formed from equilateral triangles. Show that different non-congruent polyiamonds from 1, 2, 3, 4, 5 and 6 equilateral triangles are 1, 1, 1, 3, 4 and 12 respectively. How many different non-congruent figures are made by 1, 2, 3, 4, 5, 6 regular hexagons ?

52. How many different letters of the alphabet are in each of the following classes :

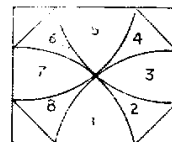
- (i) made of simple lines not enclosing a space
 (ii) enclosing one space
 (iii) enclosing two spaces.

53. Which letters of the alphabet have no axis of symmetry, one axis of symmetry, two axes of symmetry and more than two axes of symmetry ?

54. Draw the curve corresponding to the straight line in the square grid on each of the following grids.

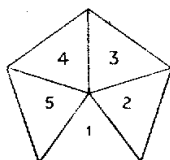


55. (a) Take a coin. Throw it 50 times, 100 times and 200 times and note in each case the number of heads and tails obtained.
 (b) Take a six-faced dice and throw it 150 times, 180 times, 210 times, 240 times, 270 times and 300 times and note in each case the relative frequencies of the number of times numbers 1, 2, 3, 4, 5 and 6 turn up.
 (c) Repeat the same experiment with each of the four other regular solids and record your result.
 (d) Rotate the octagon about a nail through O and record the frequencies with which the areas 1, 2, 3, 4, 5, 6, 7, 8 occur in the lowest position.

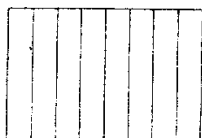


- (e) Do the same experiment with a pentagon and see what you get.

What patterns do you find in all these experiments?



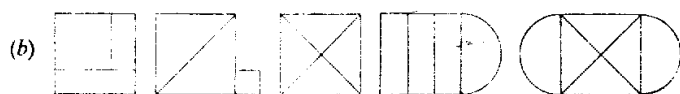
56. Take a drawing board, lined with a grid of parallel lines. Take a small needle and throw it a large number of times on the board. Notice in each case the ratio of the number of times it crosses a line to the total number of times it is thrown. What pattern do you observe?



57. Take a drawing board ruled with parallel lines and mark a spot in the centre. From a point vertically above 0 and at some distance from it, throw rice grains one by one on the board. Note the ratio of the number of grains in each rectangular grid to the total number of grains. What pattern do you observe?



58. Consider the following groups of curves:



Show that each curve of the first group can be traced without lifting the pencil, while each curve of the second group cannot be so

described. How many odd vertices (i.e., number of points where odd numbers of lines meet) are there in each curve? What conclusion can you draw? Draw more curves of both types and confirm your conclusion.

59. How many non-congruent shapes can you make with 2 cubes, 3 cubes, 4 cubes, 5 cubes, when any of two adjacent cubes has a common face?

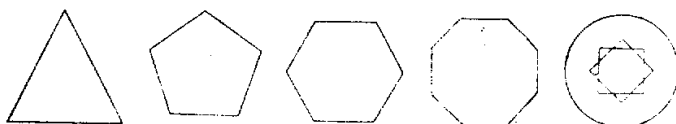
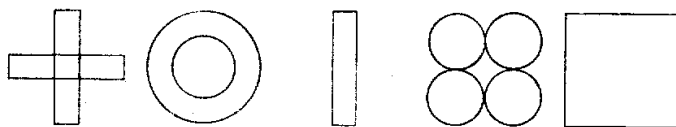
60. Make 6 of these

Assemble the six pieces together to get a cuboid of size $3 \times 4 \times 7$.

Generalise to get the formula $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

61. Take a piece of plasticine in the form of a doughnut and convert it into a tea cup.

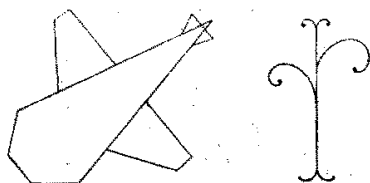
62. How many axes of symmetry has each of the following figures?



63. The following appeared in mirrors. Find the position of mirrors.

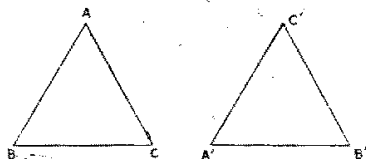
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64. Enlarge to twice the size.



65. Find $(9 \times 9) + 7$, $(9 \times 8) + 6$, $(9 \times 87) + 5$, $(9 \times 9876) + 4$, $(9 \times 98765) + 3$, $(9 \times 987654) + 2$, $(9 \times 9876543) + 1$, $(9 \times 98765432) + 0$, and comment on the pattern.

66. A triangle in a plane is displaced to some other part of the plane. Show that this displacement can be obtained as a result of at most three reflections in suitable lines in the plane.



67. Cut out card board pieces in the following shapes :

- (i) circular disc
- (ii) triangular region
- (iii) square region
- (iv) rectangular region
- (v) parallelogram region
- (vi) trapezium region
- (vii) any convex region
- (viii) any non-convex region
- (ix) two intersecting lines
- (x) three intersecting lines

Find all possible shapes of shadow of each in sunlight. What pattern do you observe? Do you find that

- (a) parallel lines in the original figure remain parallel in the shadow figure?
- (b) convex regions remain convex in the shadow figure?
- (c) intersecting lines remain intersecting in the shadow figure?

68. Show that the succession of two symmetries whose axes are at right angles is equivalent to a point symmetry or half rotation about an axis perpendicular to the plane of the axes through the point of intersection of the axes.

69. Use symmetry consideration alone to show that the diagonals of a square bisect each other.

70. Write all the 26 capital letters. Place a mirror below each and note the reflection. Which letters remain unchanged? Now similarly place the mirror above each letter or to its right or to its left and find in each case which letters remain unchanged? Classify the letters on the basis of your observations?

71. Give geometrical interpretations for the identities

$$(n+1)^2 = n^2 + n + n + 1$$

$$n^2 + n = (n+1)^2 - (n+1)$$

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

72. Multiply 12345679 by 9, 18, 27, 36, 45, 54, 63, 72, 81 and explain the pattern observed.

73. (a) The product of any three consecutive integers is always divisible by some numbers. What is the largest such number?

- (b) The product of any four consecutive integers is always divisible by some numbers. What is the largest such number?

- (c) Generalise the pattern found in (a) and (b) and verify it.

- (d) The product of any four consecutive integers when increased by unity has always a certain form. What is it?

74. Choose any three numbers less than ten and add them. Form all possible two digit numbers with these three numbers and add them. Divide the latter sum by the former. What do you get and why? Try to generalise this result.

75. Take a three digit number with all three digits different. Reverse the digits and subtract the smaller number from the larger.

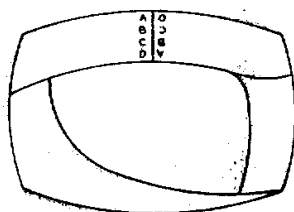
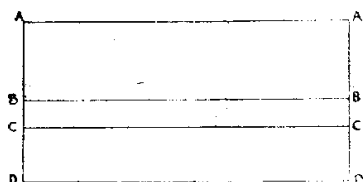
What is the middle digit and what is the sum of the other two digits? Explain. Try a similar pattern with a five digit number.

76. Take a three digit number, reverse and subtract smaller number from the larger. Again reverse the digits and add. What number do you get and why? Try the same process with a four digit number.

77. Draw a circle and draw 1, 2, 3, 4, 5, 6 straight chords in it. Find the maximum number of regions in which the circle is divided in each case and obtain the pattern.

78. Multiply two four digit numbers. Multiply (10,000—first number) by (second number—1). Add the two sums. How is this sum related to the original numbers? Generalise this to five or six digit numbers.

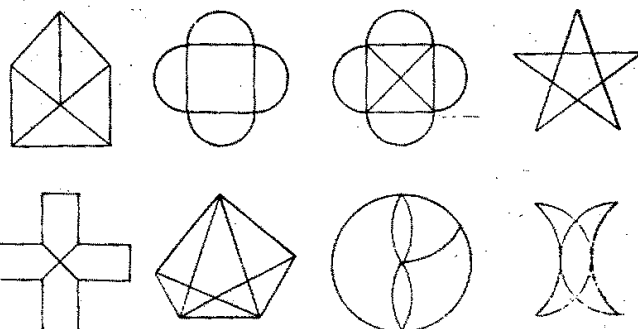
79. Take a rectangular strip as below and give it one half-twist and paste as shown to get a Mobius strip. Again make Mobius strips with 2 or 3 half twists and cut the strips along the central line or at one-third line.

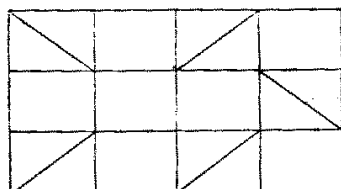
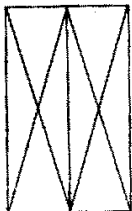


Complete the following table :

Number of half-twists	Number of sides and edges	Kind of cut	Result of cut (number of sides and edges, length and width, number of loops, twists and knots.)
0		centre	
1		centre	
1		one-third	
2		centre	
2		one-third	
3		centre	
3		one-third	

80. For each of the following networks, tabulate the numbers of odd vertices, the number of even vertices and also the result whether the network can be traversed without lifting the pencil or retracing the path. What conclusion do you draw?





Draw some more networks and check your conclusion.

81. For each of the above networks, find the number of vertices V , edges E and regions R in which the plane is divided and find $V-E+R$.

82. Obtain Pythagorean triplets with the help of the following formulae :

$$(i) \ n^2 + \left(\frac{n^2-1}{2}\right)^2 = \left(\frac{n^2+1}{2}\right)^2$$

$$(ii) \ (m^2-n^2)^2 + (2mn)^2 = (m^2+n^2)^2.$$

83. Solve the equation $a^2+b^2+c^2=r^2$ in integers when $r=3, 5, 7, 9, 11, \dots$ and tabulate all your results.

84. Draw a right-angled triangle. Draw on its three sides

- (i) squares
- (ii) rectangles with the other side of twice the length
- (iii) similar triangles
- (iv) semi circles
- (v) quarter circles
- (vi) circles inscribed in squares.

What relation do you expect between areas of the figures in each case ?

$$85. \text{ Let } x_{n+1} = \frac{1}{2} \left(x_n + \frac{N}{x_n} \right)$$

Let $N=2, x_1=1$; find $x_2, x_3, x_4, x_5, x_6, \dots$

Let $N=3, x_1=1$; find $x_2, x_3, x_4, x_5, x_6, \dots$

Let $N=5, x_1=2$; find $x_2, x_3, x_4, x_5, x_6, \dots$

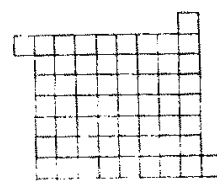
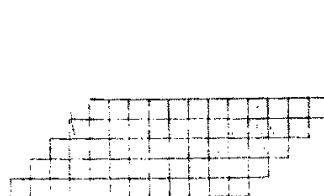
What do you find ? Explain.

$$86. \text{ Let } x_{n+1} = \frac{1}{2} \left(x_n + \frac{N}{x_n} \right)$$

and repeat the above exercise. What do you find ?

87. Fit the twelve pentominoes into

- (i) one 3×20 rectangle
- (ii) one 5×12 rectangle
- (iii) one 6×10 rectangle
- (iv) into the following shapes.



88. Use four of the pentominoes to build a 20—square pattern. Use four others to build the same pattern again. The same pattern must now be constructed a third time with the last four pieces.

89. Use twelve solid pentominoes, each constructed from five unit cubes to construct

- (i) a $3 \times 4 \times 5$ solid
- (ii) a $2 \times 5 \times 6$ solid
- (iii) a $2 \times 3 \times 10$ solid.

90. (a) Express all the numbers from 1 to 12 in terms of four ones.

(b) Multiply 1, 111, 111 by 1, 111, 111 in the usual manner and explain the identity

$$(1, 111, 111)^2 = (1, 111, 110)^2 = 1, 111, 111, 111, 111.$$

Generalise the identity.

91. Note carefully the patterns in the tables given at the next page and extend the tables by adding more numbers in each column and adding more columns.

1	2	3	6	9	18
1	2	3	6	9	18
4	5	4	7	10	19
7	8	5	8	11	20
10	11	12	15	12	21
13	14	13	16	13	22
16	17	14	17	14	23
19	20	21	24	15	24
22	23	22	25	16	25
25	26	23	26	17	26

Show that any number is equal to the sum of the numbers at the head of the columns in which it occurs. Show how this table can be prepared by using the binary scale of notation. Prepare similar tables by using scales of notation with 5 or 7 as base.

92. Prepare a set of window reading cards for the alphabet, using the binary scale of notation.

93. (a) Show that 2^{n-1} is a prime number for $n=2, 3, 7, 13, 17, 19$ and 31 .

(b) Verify that in each of the above case $2^{n-1}(2^n - 1)$ is a perfect number.

94. A number is called automorphic if all of its powers end in the same digits. Show that numbers ending in 5, 6, 25, 76, 376, 625 are automorphic. Find similar digits in other scales of notation.

95. Form tessellations with the following figures :—

(i) modified triangles



(ii) modified squares



(iii) modified hexagons.



96. Form tessellations with the following :

- (i) hexagons and triangles
- (ii) hexagons and parallelograms
- (iii) hexagons, triangles and squares
- (iv) octagons and squares
- (v) dodacagons and triangles.

97. (a) A boatman can carry a wolf, a goat and a cabbage across a river in a boat so small that he can take only one at a time. Furthermore, he must be on hand to keep the wolf from eating the goat and the goat from eating the cabbage. How shall he do it?

(b) Two jealous husbands and their wives must cross a river on a boat that holds only two persons. How can this be done so that a wife is never left with the other woman's husband unless her own husband is present.

98. (a) There are twelve coins, one of which is known to be heavier than the others. What is the minimum number of weighings with an ordinary balance in which you can find the defective coin?

(b) Generalise the problem to any number of coins.

99. There are three men A, B, C, and three jobs I, II, III. In how many ways can one job be assigned to each man? Generalise the problem to any number n of men and n of jobs.

100. There are 3, 4, 5, 6, 7, 8 wagons available at stations A, B, C, D, E, F. Wagons required at stations I, II, III, IV are 6, 7, 8, 10. Suggest at least ten ways of transporting the wagons.

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